Adam Siegel
Senior Capstone
Dr. Papantonopoulou
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REGULAR SOLIDS
and their Rotational Groups

I. Introduction

The *symmetry group* for a physical object is the set of ways that object can be repositioned so that it maintains its symmetry, or looks the same. Any object of any dimension has such a group of symmetries. Some objects, like the letter J, have only one symmetry called the *identity* because the only way they maintain their symmetry is to remain in the same place. Other objects, like an equilateral triangle, have several symmetries because they can be repositioned by rotations or reflections and still maintain symmetry.

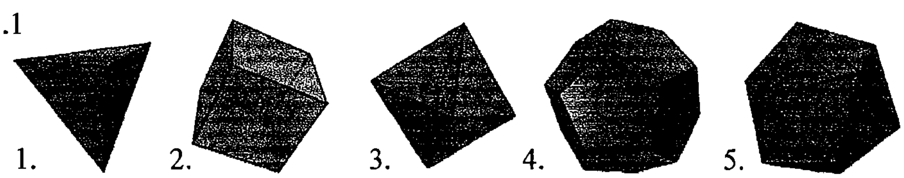
The symmetries of any object form a mathematical *group* under symmetry. The set of three-dimensional objects called *regular solids* possess many beautiful symmetries and accordingly form interesting symmetry groups under their rotations.

A *Regular Solid*, or Platonic solid, is a three-dimensional polyhedron such that each face is a regular polygon, all faces are isometric to each other, and all vertices are isometric to each other. For geometrical objects, *isometric* means congruent. Therefore every face of a regular solid has the same number of edges and every vertex connects to the same number of faces.

It turns out that among all the three-dimensional objects that can possibly be created, only five regular solids can be formed. Each is named for its number of faces. The five regular solids are:

1. The *Tetrahedron*, a 4-sided solid with equilateral triangles as faces
2. The *Hexahedron* or *Cube*, a 6-sided solid with squares as faces
3. The *Octahedron*, an 8-sided solid with equilateral triangles as faces
4. The *Dodecahedron*, a 12-sided solid with regular pentagons as faces
5. The *Icosahedron*, a 20-sided solid with equilateral triangles as faces

Figure 1.1



Any regular or non-regular *polyhedron*, which is any completely enclosed solid with regular or non-regular polygonal faces, has v vertices, e edges, and f faces. *Euler's Observation* tells us that any polyhedron maintains the equality $f+v-e=2$, which we note holds true for the regular solids in Table 1.2.

For the regular solids, we know that each vertex is isometric to every other vertex. Therefore the same number of faces k meet at each vertex. We also know each face must be the same regular polygon, so each face as n sides. See Table 1.2 to see the properties of each regular solid.

Table 1.2

Regular Solid	n	k	v	e	f
Tetrahedron	3	3	4	6	4
Cube	4	3	8	12	6
Octahedron	3	4	6	12	8
Dodecahedron	5	3	20	30	12
Icosahedron	3	5	12	30	20

Because of the symmetry created by a regular solid, we can find some equalities among the values of *v*, *e*, *f*, *k*, and *n*.

Note that a regular *n*-gon has *n* vertices which implies each face of a regular solid touches *n* vertices. If we multiply the vertices of each face *n* by the number of faces *f*, we get a multiple of the number of vertices of the entire regular solid. We do not get the exact number *v* because each vertex is touching more than one face – each vertex is touching *k* faces. Therefore multiplying $n \cdot f$ will give us *k* times the number of vertices *v*. Hence, for all regular solids $n \cdot f = k \cdot v$.

We can also count the edges in a similar manner. Each edge has one vertex on each end, so each edge touches two vertices. Therefore multiplying 2 by *e* will give us a multiple of the amount of vertices of the regular solid. Since each vertex touches *k* faces, it is also touching *k* edges. Hence the multiple of *v* will be $k \cdot v$ once again, and we conclude $2 \cdot e = k \cdot v = n \cdot f$.

II. Proving How Many Regular Solids Exist

It seems amazing that among all the 3-dimensional objects, only five regular solids exist. However we can prove that there are only five.

Theorem: Only five regular solids exist.

Proof: We found that any regular solid has the equality $2 \cdot e = k \cdot v = n \cdot f$.

Euler's Observation states $f + v - e = 2$. Combining the equalities we get:

$$\frac{2 \cdot e}{k} + \frac{2 \cdot e}{n} - e = 2 \Rightarrow \frac{2en}{kn} + \frac{2ek}{kn} - \frac{ekn}{kn} = 2 \Rightarrow 2en + 2ek - ekn = 2kn$$

$\Rightarrow \frac{1}{2enk}(2en + 2ek - ekn) = \frac{1}{2enk}(2kn) \Rightarrow \frac{1}{k} + \frac{1}{n} - \frac{1}{2} = \frac{1}{e}$. Note $\frac{1}{e} > 0$ since no polyhedron can have 0 or fewer edges. Thus $\frac{1}{k} + \frac{1}{n} - \frac{1}{2} = \frac{1}{e} > 0 \Rightarrow \frac{1}{k} + \frac{1}{n} - \frac{1}{2} > 0$

$\Rightarrow \frac{1}{k} + \frac{1}{n} > \frac{1}{2}$. Thus, any *k* and *n* combination where $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$ will form a regular solid.

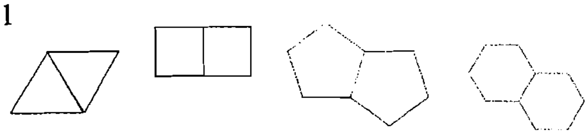
Before we find such combinations, we must note the following restrictions:

i.) First we note that $n \geq 3$ since polygons must have at least 3 sides.

ii.) Also note that $k \geq 3$ in all cases. Clearly if $k = 1$, then the entire figure would be one sided, and hence two-dimensional. If all vertices only touched one side, there would only be one side. If $k = 2$, then two adjacent faces would be forced to lie in the same plane in order to keep the solid completely enclosed. Likewise the object would reduce to two dimensions.

See Figure 2.1 for an illustration of this contradiction for 3, 4, 5, and 6 sided polygons, confirming $k > 2$ in all cases. Notice how if any of these sets of polygons were folded so that edges meet and form an enclosure, it would force adjacent faces to overlap.

Figure 2.1



iii.) Note that for any *n*, once we find some $\hat{k}$ such that the combination of *n* and $\hat{k}$ do not form a regular solid, then the combination of *n* and *k* for all $k > \hat{k}$ will also fail to form a regular solid, by the following: Given $k > \hat{k}$ where $\frac{1}{\hat{k}} + \frac{1}{n} \le \frac{1}{2}$, it follows that

$\hat{k} < k \Rightarrow \frac{1}{k} < \frac{1}{\hat{k}} \Rightarrow \frac{1}{k} + \frac{1}{n} < \frac{1}{\hat{k}} + \frac{1}{n} \le \frac{1}{2}$.

iv.) Similarly, once we find some $\hat{n}$ where we cannot form a regular solid for any *k*, then we cannot form a regular solid for any $n > \hat{n}$, by the following: Given $n > \hat{n}$ where $\frac{1}{k} + \frac{1}{\hat{n}} \le \frac{1}{2}$ for all *k*. If it's true for all *k*, then it's true for $k = 3$. Thus $\frac{1}{3} + \frac{1}{\hat{n}} \le \frac{1}{2}$.

$\hat{n} < n \Rightarrow \frac{1}{n} < \frac{1}{\hat{n}} \Rightarrow \frac{1}{3} + \frac{1}{n} < \frac{1}{3} + \frac{1}{\hat{n}} \le \frac{1}{2}$. Thus for all $n > \hat{n}$, *n* and $k = 3$ do not form a regular solid. In note iii. above, we showed if *n* and $k = 3$ do not form a regular solid, then *n* and all $k > 3$ do not form a regular solid. Since $k \ge 3$ in all cases then all $n > \hat{n}$ cannot form a regular solid with any possible *k*.

With these restrictions, we must only inspect a finite amount of combinations of *n* and *k* to find which combinations form regular solids. As stated above, for any *n* once we find a *k* that doesn't form a regular solid we can stop. Likewise once we find an *n* that cannot create a regular solid we can stop. See Table 2.2 where we check all possible combinations and show only five such combinations form regular solids.

Table 2.2

n	k	$\frac{1}{k} + \frac{1}{n}$	Is $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$?	Regular Solid Formed
3	3	2 / 3	YES	Tetrahedron
3	4	7 / 12	YES	Octahedron
3	5	8 / 15	YES	Icosahedron
3	6	1 / 2	NO	
4	3	7 / 12	YES	Cube
4	4	1 / 2	NO	
5	3	8 / 15	YES	Dodecahedron
5	4	9 / 20	NO	
6	3	1 / 2	NO	

Hence we have proven there are exactly five regular solids that can be formed. □

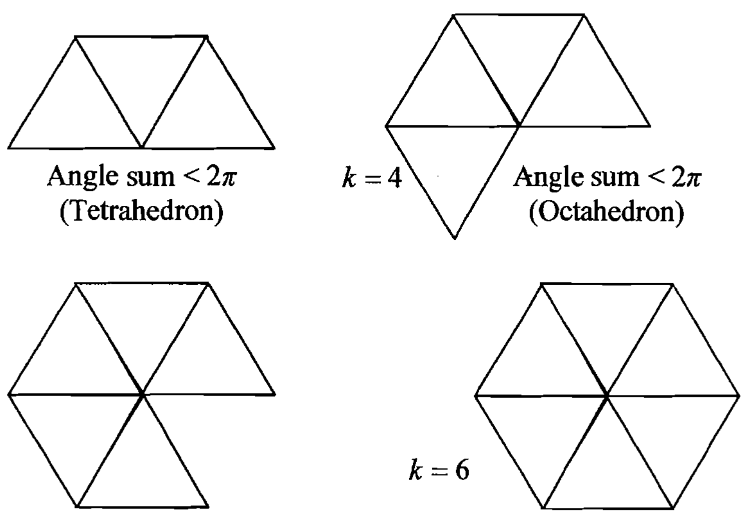
Finding such possible combinations can also be illustrated quite nicely. Since each vertex is isometric to each other, we may inspect one vertex and decide whether it can be a vertex on regular solid with a given n and k.

We can represent a vertex in two dimensions by "flattening" the vertex so that we see all of its neighboring faces in one plane. To visualize the regular solid, imagine the polygons being folded into the paper until the edges of all polygons meet and begin to enclose the solid. The vertex representations that can form a regular solid are the ones that can be folded into the paper in such a way. If the edges already meet or overlap, then the object cannot be folded into the paper and will be stuck in two dimensions, thus failing to form a regular solid.

This implies the condition that the interior angles of the faces meeting at a vertex must not be greater than 2π. Since an interior angle of an n-gon is $\pi - 2\pi / n$, the sum of the interior angles of k faces is $k[\pi - 2\pi / n]$. Thus in order to form a regular solid, we must have a k and n such that $k[\pi - 2\pi / n] < 2\pi$ (Pap. 466).

In Figures 2.3-2.5 we see the vertex representations for the regular solids, as well as vertex representations of n and k combinations that do not form regular solids by having the sum of the interior angles greater than or equal to 2π. Compare the n and k combinations in the following Figures to Table 2.2 to see that both methods find the same combinations of n and k to form regular solids.

Figure 2.3, $n = 3$:



$k = 3$ Angle sum $< 2\pi$ (Tetrahedron)

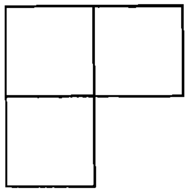
$k = 4$ Angle sum $< 2\pi$ (Octahedron)

$k = 5$ Angle sum $< 2\pi$ (Icosahedron)

$k = 6$ Angle sum $= 2\pi$ (NOT a regular solid)

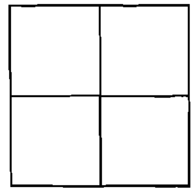
Figure 2.4, $n = 4$:

$k = 3$



Angle sum $< 2\pi$
(Cube)

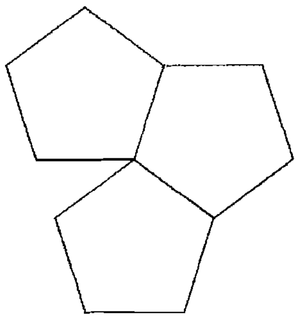
$k = 4$



Angle sum $= 2\pi$
(NOT a regular solid)

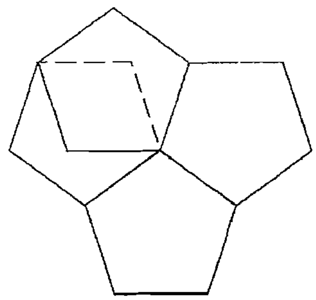
Figure 2.5, $n = 5$:

$k = 3$



Angle sum $< 2\pi$
(Dodecahedron)

$k = 4$

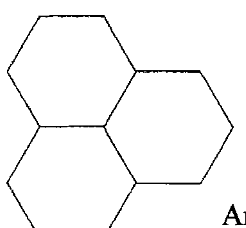


Angle sum $> 2\pi$
(NOT a regular solid)

The above condition for the sum of interior angles also tells us that no regular solid can be formed for polygons of six or more sides. See Figure 2.6 which shows a vertex joining three regular hexagons. Since the edges already meet, the figure is trapped in two-dimensions. This would form an interesting tessellation or floor tile, but not a regular solid. Also since any n-gon where $n > 6$ would have larger interior angles than the hexagon, this confirms that $n \leq 5$ in order to form a regular solid.

Figure 2.6, $n = 6$:

$k = 3$



Angle sum $= 2\pi$
(NOT a regular solid)

Recall for the objects in Figures 2.3-2.5, the description instructed the reader to "imagine the polygons being folded into the paper until the edges all polygons meet and begin to enclose the solid". If we continue forming isometric vertices from that first vertex, we could create the entire regular solid. Since any one vertex is isometric to every other vertex, knowing the properties of one vertex can generate the entire solid in this fashion. Therefore all we need to know are the values of *n* and *k* to form a regular solid.

III. Symmetry Group Axioms

The beautiful thing about the regular solids is their symmetry. Since all vertices, faces, angles, and edges are isometric to each other, each solid can be rotated in many ways while maintaining all these symmetries. Thus the rotations of each regular solid that maintain all symmetries form a mathematical group under rotation.

We will refer to such groups as G_S, where $S=\{T,C,O,D,I\}$ referring to the first letter of each regular solid. The following is an informal proof that the rotations of the regular solids possess the four group axioms:

i.) Associative: Since rotating any solid permutes the vertices *v*, then $G_S \subseteq S_v$. Since S_v is associative, so is its subset G_S.

ii.) Identity: If we simply do not move the solid, everything remains fixed. Therefore the identity is when we do not rotate the solid at all.

iii.) Closure: If we follow any rotation that maintains all symmetries with a second rotation that maintains all symmetries, then clearly all symmetries are maintained and we are still inside the group of rotations that maintain symmetries.

iv.) Inverse: For any rotation that takes the solid to some new position, we can simply rotate it back to its original position which would be the inverse rotation.

IV. Symmetry Group Orders

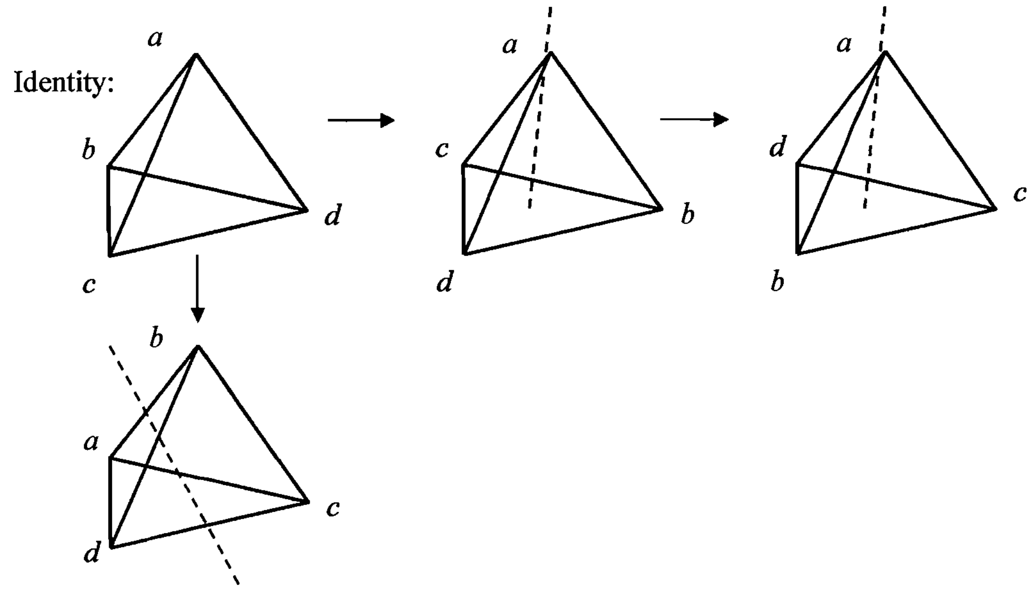
There are several ways to find the order of the group of rotations G_S of each regular solid. Knowing the order will help us classify the rotational groups later.

Since the order of the group is equal to the number of elements in the group, or the number of rotational symmetries of each regular solid, we can simply count how many ways we can rotate each solid so that it maintains a similar position. The tetrahedron is the easiest to visualize with this method.

The first element of the group of rotations of the tetrahedron is the identity, not moving the solid at all. Additionally, the tetrahedron can be rotated by $2\pi/3$ or $4\pi/3$ while fixing each of the four vertices, which accounts for 8 more elements. Also we can swap any pair of vertices, but notice doing so forces us to swap the remaining two vertices as well or else we would have a twisted or inverted solid. This will account for three such elements, for the three sets of pairs of vertices. This rotation is a rotation around the line connecting the midpoint of edge *ab* and the midpoint of edge *cd*. See Figure 4.1 for an illustration of these elements.

This gives us 12 elements in total, thus telling us the order of the group of rotations of a tetrahedron is 12. By the description of these elements we recognize this group to be isometric to A_4, the group of even permutations of four elements. We will prove this later on. For now we are just noting that $|G_T|=12$.

Figure 4.1



This method for finding the order is manageable for the tetrahedron, but would become more cumbersome for the other solids. For example, it would be terribly difficult to find all the permutations of vertices for the 20 vertices of the dodecahedron. There are in fact easier methods for simply finding the order and we can do this by using the orbit-stabilizer theorem.

The following three methods to finding the order uses the condition we found earlier for all regular solids: $k \cdot v = n \cdot f = 2 \cdot e$. This formula is actually three representations of the orbit-stabilizer theorem and all three formulas are equal to the order of the group $|G_S|$.

Recall: For group G acting on set X by permutation so that $G \cdot X \to X$. The *stabilizer* G_a for some $a \in X$ is the group of elements in G that fix a. The *orbit* O_a for some $a \in X$ are all $b \in X$ where $b = g \cdot a$ for some $g \in G$. The Orbit-Stabilizer Theorem states that $|G_a| \cdot |O_a| = |G|$ (Pap. 139).

1.) Using vertices: In this method we fix a vertex a and find how many possible rotations there are that fix that vertex. This is the same thing as finding the order of G_a, the stabilizer of a. Since vertex a touches k faces and any rotation must send a face to a face, then we can rotate each solid in k different ways while keeping vertex a fixed, i.e. $|G_a| = k$.

Then we find how many places to which vertex a can be rotated. Since each vertex is isometric to every other vertex, vertex a can go to any of the v vertices. This is the orbit of a, so we have $|O_a| = v$. By the orbit-stabilizer theorem this tells us the order of the group of rotations is found by the formula: $|G_S| = k \cdot v$.

2.) Using faces: In this method we fix some face *b* on the solid and then find how many ways we can rotate the solid so that the fixed face stays in the same place. This is the order of G_b, the stabilizer of *b*. This is equal to *n*, the number of sides on each face of the solid since we can rotate every face *n* ways while maintaining symmetry, i.e. $|G_b| = n$.

Then we find how many places to which face *b* can be rotated. Since each face is isometric to every other face, face *b* can go to any of the *f* faces. This is the orbit of *b*, and we find $|O_b| = f$. By the orbit-stabilizer theorem this tells us the order of the group of rotations is found by the formula: $|G_s| = n \cdot f$

3.) Using edges: Lastly, fix any edge *c* of the solid. Each edge can stay fixed for exactly two rotations for any solid, since the endpoints of each edge can switch places with each other. Thus for every edge, the stabilizer is only two elements, i.e. $|G_c| = 2$.

Then we find how many places to which edge *c* can be rotated. Since each edge is isometric to every other edge, edge *c* can go to any of the *e* edges. This is the orbit of *c*, so $|O_c| = e$. Finally, the orbit-stabilizer theorem tells us the order of the group rotations is found by the formula $|G_s| = 2 \cdot e$.

In summary, the order of the group of rotations of any regular solid has the equality: $|G_S| = k \cdot v = n \cdot f = 2 \cdot e$.

Keeping this formula in mind, we see reproduce the earlier table in Table 4.2 with a new column for the order of the rotational group for each solid. We now notice some interesting equalities in the properties of the regular solids.

Table 4.2

Regular Solid	***k***	***v***	***n***	***f***	***e***	$\lvert G_s \rvert$
Tetrahedron	3	4	3	4	6	12
Cube	3	8	4	6	12	24
Octahedron	4	6	3	8	12	24
Dodecahedron	3	20	5	12	30	60
Icosahedron	5	12	3	20	30	60

From the table above we clearly see that $|G_S| = k \cdot v = n \cdot f = 2 \cdot e$ for all of the regular solids.

We also notice two interesting properties. First that each regular solid's rotation group G_i has the same order as that of another regular solid G_j. We also see that for every regular solid *i*, that both $k_i = n_j$ and $v_i = f_j$ for that solid *j*. Specifically, these equalities match the cube and the octahedron, the dodecahedron and the icosahedron, and the tetrahedron to itself.

These regular solids that have such a match turn out to be regular solids that are *dual* to each other as we will find later.

V. Spheres

The regular solids have some interesting geometrical properties in that each regular solid can have:

a.) A *circumscribed sphere* through its vertices so that each vertex of the solid tangentially touches the sphere

b.) A *midsphere* where the sphere tangentially touches the midpoint of each edge on the solid

c.) An *inscribed sphere* where the sphere tangentially touches the midpoints of each face on the solid

Due to the symmetry of the regular solids and the isometric property between all vertices, edges, and faces, we see that each vertex is equidistant from the center point of the regular solid and likewise for the midpoints of the edges and the midpoints of the faces. This is the basis of being able to rotate the regular solids sending vertices to vertices, edges to edges, and faces to faces.

Additionally, each of these spheres creates more symmetry inside each regular solid. For two of the spheres described above, connecting all the points where the sphere and regular solid intersect will create another regular solid where the vertices are the intersecting points. In the midsphere connecting the points where the regular solid intersects the sphere creates an Archimedean solid.

a.) For the circumscribed sphere, it is clear that the "new" regular solid created is exactly the same as the old one. Since the intersection is the set of vertices of the regular solid, clearly they are just forming the same set of vertices.

b.) Since the intersection of the midsphere and the regular solid connects the midpoints of each edge, we are truncating the regular solids. This creates an *Archimedean solid,* which is a polyhedron where all faces are regular polygons which may or may not be the same, all vertices are isometric to each other, and all edges have equal length. Note that regular solids are a subset of the Archimedean solids. The Archimedean solid created by the midsphere in the tetrahedron is the octahedron, the solid created inside a cube or octahedron is the *cuboctahedron,* and the solid created inside a dodecahedron and icosahedron is the *icosidodecahedron.* See these Archimedean solids in Figure 5.1. Note the "matching" between the cube and octahedron as well as that between the dodecahedron and icosahedron are again seen here.

Figure 5.1

cuboctahedron

icosidodecahedron

c.) The inscribed sphere is perhaps the most interesting. Since the set of vertices on the new solid *i* is the set of midpoints of the faces of the original solid *j*, then the number of vertices on the new solid v_i is equal to the number of faces on the original solid f_j, i.e. $v_i = f_j$. Likewise, each new vertex on *i* will originate from some face on *j* and connect to k_i vertices. Since each vertex will connect to the midpoint of all neighboring faces to the originating face on *j*, it will connect to n_j faces. Therefore each new vertex on *i* will have an edge connecting to n_j vertices and we have $k_i = n_j$.

The two equalities created by the inscribed sphere, $v_i = f_j$ and $k_i = n_j$, tell us the exact identity of the new regular solid. Recall from Table 4.2, each regular solid *i* had a "matching" regular solid so that both $v_i = f_j$ and $k_i = n_j$. Thus whatever regular solid we chose for *i*, we will know which regular solid *j* must be immediately.

VI. Duals

The new regular solid created by the inscribed sphere is called the *dual* solid of the original. Dual solids possess the characteristic of having complimentary numbers of vertices and faces, which is immediate from the description of its construction.

We also see that the dual of a regular solid's dual is the original regular solid. For example, a cube inscribed inside an octahedron can have its own dual, an octahedron, inscribed inside. Accordingly we can create an infinite chain of duals inside of each other.

See Figure 6.1 to see how the cube and octahedron are dual to each other. Figure 6.2 shows an example of the duals inscribed in each solid.

Figure 6.1

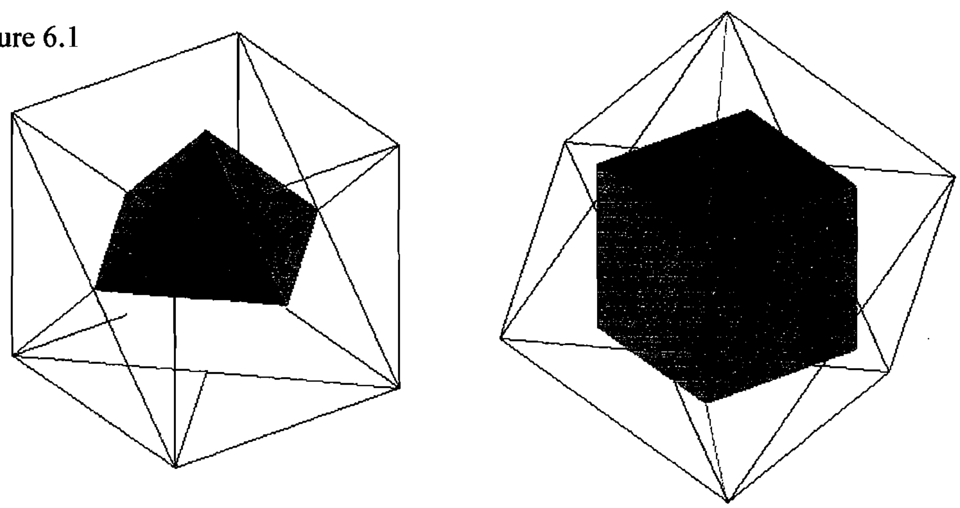
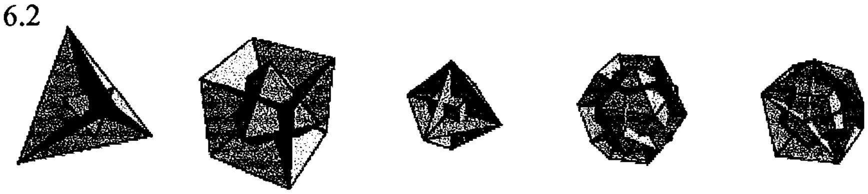


Figure 6.2



Duality is very important as we next classify the groups of rotations of each solid because the groups of rotations of dual solids are isomorphic. Since the dual of a solid is created from midpoints, it is perfectly symmetrical inside of the original solid. Therefore when the original solid is rotated in any manner, its dual is rotated in the exact same way. In fact, if you created a new object that was a regular solid with its dual constructed inside like in Figure 6.1, it would have all the same symmetries as either solid on its own.

Therefore we know $G_i \cong G_j$ for dual regular solids *i,j*. Specifically: $G_C \cong G_O$, $G_D \cong G_I$, and obviously $G_T \cong G_T$.

VII. Classifying the Groups

Now that we have determined which regular solids are dual to each other, we will have two less groups to find. We will now classify:

1.) The groups of rotations of the tetrahedron G_T.
2.) The groups of rotations of the cube G_C, which will also give us the group of rotations of its dual, the octahedron, G_O.
3.) The groups of rotations of the dodecahedron G_D, which will also give us the group of rotations of its dual, the icosahedron, G_I.

To do so we will use the fact that each group of rotations acts on a set of features of the regular solids by permutation. The set of features may be, for example, the set of vertices or diagonals. We know this is a group action on such elements since all rotations of each regular solid must maintain all symmetries. Therefore each rotation mapping vertices to vertices is in actuality permuting the vertices.

In this approach it is necessary to find a set of features of the regular solids X where the rotational group of symmetries acts faithfully on X. Recall G acts *faithfully* on X if the only element of G that fixes every element of X is the identity. Lastly note that the group denoted S_n refers to the permutation group permuting n elements.

In classifying the groups of rotations for the regular solids, we will use the following theorem:

Theorem 7: If the group of rotations of a regular solid G_S acts faithfully on some set of features of the regular solid X, then G_S is a subgroup of S_X.

Proof: Let G_S be the rotational group of symmetry for some regular solid. Let X be some set of features of the regular solid, i.e. vertices, diagonals, or inscribed cubes. Assume G_S acts on X faithfully, which means the only element of G_S that fixes every element of X is the identity.

Construct a homomorphism $\phi: G_S \to S_X$. We know this is a good map since we are permuting the elements of X. Consider the image of $\phi(G_S)$. We know the image is a subgroup of S_X, i.e. $\phi(G_S) \leq S_X$.

By the first isomorphism theorem, $G_S / \text{Kern}\phi \cong \phi(G_S)$. But if G_S acts on X faithfully, then the kernel is trivial. This implies $G_S / \text{Kern}\phi \cong G_S \cong \phi(G_S) \leq S_X$. Thus, $G_S \leq S_X$.

$\square$

Using this theorem, we can now classify the rotational group of symmetry for each regular solid.

The Tetrahedron:
When we were finding the elements of the tetrahedron, we discovered the elements seemed to form A_4, the group of even permutations permuting four elements. Using the previous theorem, we can now prove that $G_T \cong A_4$.

Theorem: The group of rotations of the tetrahedron is isomorphic to A_4, i.e. $G_T \cong A_4$.

Proof: Let G_T act on X, the set of the four vertices of the tetrahedron. Since the only element of G_T that fixes every vertex is the identity, then we know G_T acts faithfully on X. Therefore by Theorem 7, we know $G_T \leq S_4$. Since $|S_4| = 4! = 24$ and we found earlier that $|G_T| = 12$, then G_T must be the only normal subgroup of order 12 in S_4, which is A_4 the group of even permutations in S_4. Therefore $G_T \cong A_4$. $\square$

We may note two other ways we could have determined $G_T \cong A_4$. For one, we can easily find four distinct 3-cycles in G_T permuting four elements, the vertices. We find these by rotating the tetrahedron by 0, $2\pi/3$, and $4\pi/3$ while fixing each of the four vertices. Such three cycles generate the group A_4. This would prove $A_4 \leq G_T$ $\Rightarrow A_4 \cong G_T$ since $|A_4| = |G_T|$.

We may also note that A_4 is the only group of order 12 where the element of maximal order is three. We can determine that the maximal order of an element in G_T is three since it has triangular faces and three faces meeting at any one vertex. Any rotation will reach the identity on or before the third rotation. Then once we found the order using one of the methods described earlier, the fact $G_T \cong A_4$ is immediate.

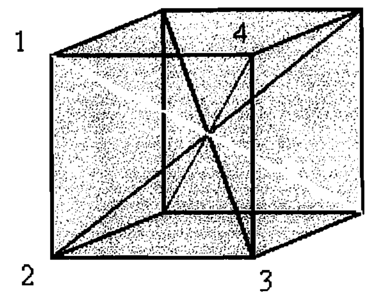
The Cube:

We will now determine to which group the rotations of the cube, G_C, is isomorphic. Similarly to our approach to the tetrahedron we will use Theorem 7 and the fact that G_C acts faithfully on some set X.

For the tetrahedron, we defined X as the set of vertices of the tetrahedron. It is true that G_C acts faithfully on the set of vertices in the cube, since the only rotation that fixes all eight vertices is the identity. However this will only tell us that G_C sits inside S_8, a group of order 40,320 and not such an easy group to work with. Therefore to classify G_C, we must find a smaller set X on which G_C acts faithfully.

A smaller set X on which G_C acts faithfully is the set of diagonals, the lines connecting opposite corners of the cube as seen in Figure 7.1. We know that G_C acts faithfully because the only rotation of the cube that fixes all four diagonals is the identity rotation, not moving the cube at all.

Figure 7.1



Theorem: The group of rotations of the cube is isomorphic to S_4, i.e. $G_C \cong S_4$.

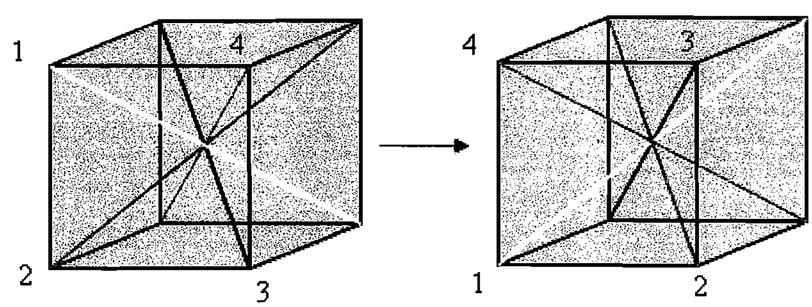
Proof: Let G_C act on X, the set of the four diagonals of the cube. Since the only element of G_C that fixes every diagonal is the identity, then G_C acts faithfully on X. Therefore by Theorem 7, we know $G_C \leq S_4$. Since $|S_4| = 4! = 24$ and we found earlier that $|G_C| = 24$, then G_C must be the improper subgroup of S_4. Thus, $G_C \cong S_4$.

□

Once again note that since the cube and the octahedron are dual solids, that this also implies $G_O \cong S_4$ as well.

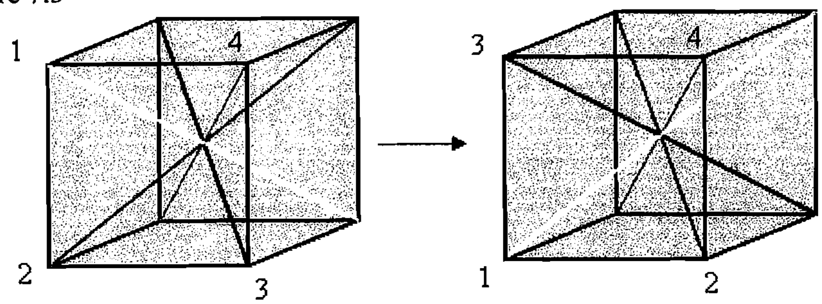
We can find all elements and subgroups of S_4 by looking at the actions certain rotations take on the diagonals. For example, a 4-cycle in S_4 is any $\pi/2$ rotation along an axis through the midpoints of opposite faces. In Figure 7.2, this is the permutation (1234). Note this is also an example of how Z_4 is a subgroup of S_4 since $|(1234)| = 4$.

Figure 7.2



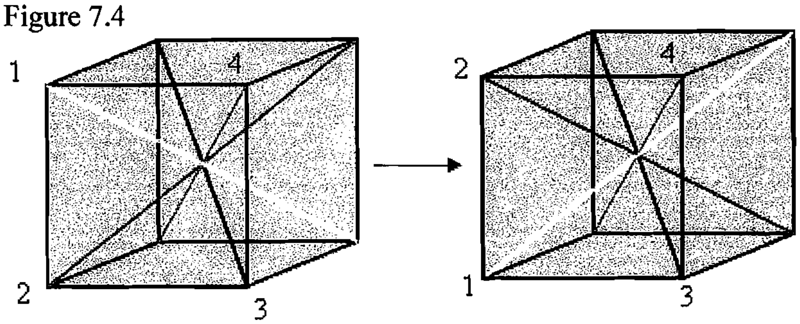
We can also find any 3-cycle by rotating the cube along any diagonal, as in Figure 7.3 which shows the permutation (123) as we rotate along the blue diagonal labeled "4". This is also an example of the subgroup Z_3 that sits inside S_4.

Figure 7.3



We can also find any 2-cycle by "fixing" any two diagonals and rotating the cube so that the remaining two diagonals switch places. See Figure 7.4 for an illustration of permutation (12). Note that to create a cycle of form $(ab)(cd)$, we may perform this step for two distinct pairs of diagonals simultaneously. Such an element is an example of the subgroup Z_2 that sits inside S_4.

Figure 7.4



We showed we can find the cyclic subgroups Z_2, Z_3, and Z_4 as in Figure 7.2-7.4. Also $\mathrm{Z}_2 \times \mathrm{Z}_2$ is generated by any two distinct rotations that are isomorphic to Z_2 as in Figure 7.4.

We can also find the non-Abelian groups S_3, D_4, and A_4. The group S_3 is generated by performing the actions of S_3 on the three pairs of opposite faces as seen in Figure 7.5. We can note the importance of faithfulness in Theorem 7 from these pairs of opposite faces. We see G_C is not faithful on the set of opposite faces since, in addition to the identity, a rotation of $\pi/2$ in any direction fixes all opposite faces. Therefore since faithfulness fails in this instance, we cannot use Theorem 7 to say G_C is a subgroup of S_3.

Also note, the dihedral group D_4 is generated by the rotations fixing one pair of opposite faces as seen in Figure 7.6. Lastly, A_4 is generated by the tetrahedron created inside the cube, as we found that the rotations of the tetrahedron are isomorphic to A_4. This can be seen in Figure 7.7.

Figure 7.5

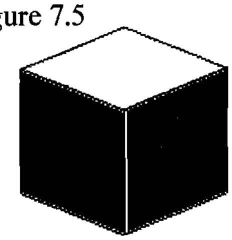


Figure 7.6

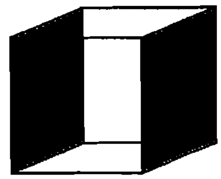
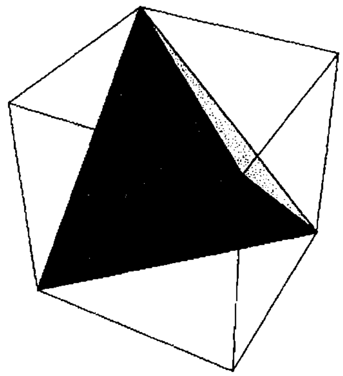


Figure 7.7



Lastly, we may note we could have also determined that $G_C \cong S_4$ once we found that there were elements of G_C permuting the diagonals by (1234) and (12) as we did in Figure 7.2 and 7.4. Since (1234) and (12) generate S_4, then we could prove $S_4 \leq G_C$ $\Rightarrow S_4 \cong G_C$ since $|S_4| = |G_C|$.

The Dodecahedron:

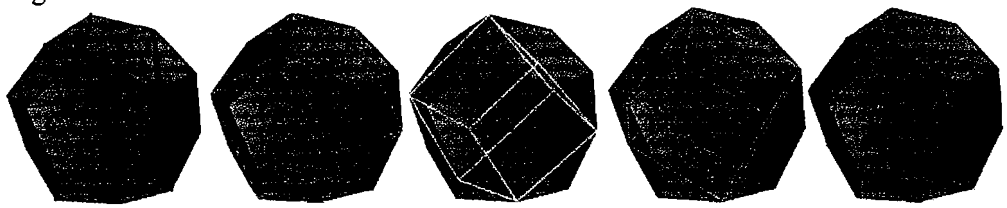
Similar to the cube, we will not classify G_D by using the fact the group acts faithfully on the set of vertices since there are twenty of them. In this case even the diagonals are hard to deal with since there are ten diagonals. However, the dodecahedron does possess another smaller set of elements that the rotations in G_D permute. It turns out that amongst the dodecahedron's symmetry, five distinct cubes can be inscribed inside any dodecahedron.

Take the diagonal of a pentagon on any face of the solid. We can construct two more orthogonal lines from the original diagonal so that they are diagonals on neighboring faces. This forms a vertex and three edges of a cube. Continue forming orthogonal edges that are also diagonals of the pentagon faces until eventually an entire cube is formed inside the dodecahedron.

We know this figure we constructed is a cube because all edges of the figure are diagonals of a pentagon which are parallel to an edge of the pentagon. Opposite edges of the figure are parallel to the same edge of some pentagon and hence parallel to each other. This implies all faces of the figure are parallelograms. Because of the symmetry of the dodecahedron, both diagonals of each parallelogram are equal which implies each face is a rectangle. Since all edges of the rectangles are equal, then the faces are squares. Since each face of the figure is a square then the figure is a cube.

Since the cube has twelve edges and the dodecahedron has twelve faces, then some edge of the cube will be the diagonal on each face of the dodecahedron. Since each pentagon face contains five diagonals, then we can construct five such inscribed cubes. See Figure 7.8 for an illustration of the five such cubes.

Figure 7.8



The group of rotations of the dodecahedron, G_D, permute the position of these five inscribed cubes and the only rotation that fixes all five cubes is the identity. We know this is true since, as stated earlier, one edge from each of the five cubes is the diagonal on each face of twelve faces. The only way for all five diagonals of a pentagonal face to remain fixed is for the entire dodecahedron to be unmoved. Therefore the only element of G_D that fixes all five inscribed cubes is the identity, which means G_D acts faithfully on the five inscribed cubes.

Finding this relatively small set of elements on which G_D acts faithfully allows us to use Theorem 7 to prove $G_D \cong A_5$.

Theorem: The group of rotations of the dodecahedron is isomorphic to A_5, i.e. $G_D \cong A_5$.

Proof: Let G_D act on X, the set of the five inscribed cubes of the dodecahedron as in Figure 7.8. Since the only element of G_D that fixes all five cubes is the identity, then we know G_D acts faithfully on X. Therefore by Theorem 7, we know $G_D \leq S_5$. Since $|S_5| = 5! = 120$ and we found earlier that $|G_D| = 60$, then G_D must be the only normal subgroup of order 60 in S_5, which is A_5 the group of even permutations in S_5. Therefore $G_D \cong A_5$. □

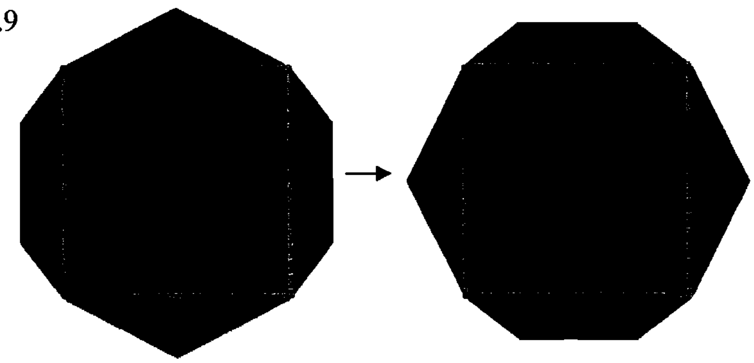
Once again note that since the dodecahedron and the icosahedron are dual solids, that this also implies $G_I \cong A_5$ as well.

Finding a cube inside the dodecahedron immediately could make one think that S_4 sits inside G_D since we found that the group of rotations of the cube is isomorphic to S_4. After all, we found the tetrahedron sitting inside the cube and we confirmed that the rotations of a tetrahedron are a subgroup of the rotations of a cube since $A_4 \leq S_4$.

However this cannot be true for a cube inside a dodecahedron. Recall we found earlier that $|G_D| = 60$ and we know $|S_4| = 24$. Since 24 does not divide 60, then S_4 cannot be a subgroup of G_D as it would contradict LaGrange's Theorem.

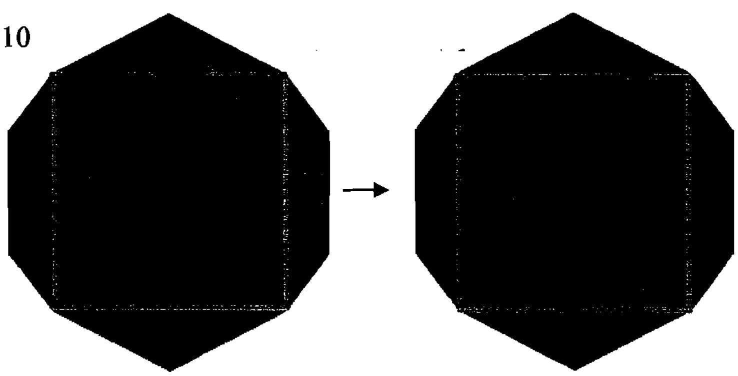
The difference is the rotations of a cube inside a dodecahedron do not always maintain symmetry in the dodecahedron. For example, rotating the cube by $\pi/2$ (or by element $(1234) \in S_4$) does not maintain symmetry of the outside dodecahedron. See Figure 7.9 for an example. Notice how the rotation of the cube alters the symmetry of the dodecahedron so that it is no longer an element of G_D.

Figure 7.9



Therefore we can definitively say S_4 is not a subgroup of G_D. However, some elements in S_4 must sit inside G_D. There are some rotations of the cube that maintain all the symmetries of the dodecahedron. For example, if we rotated the cube by π (or by element $(13)(24) \in S_4$) we stay within the elements of the dodecahedron, as seen in Figure 7.10.

Figure 7.10



Notice a rotation by π is an even permutation whereas a rotation by $\pi/2$ is not. It turns out the only rotations of the cube that maintain the symmetries of the dodecahedron are the even permutations of the cube. We know the rotations of the cube are isomorphic to S_4, and we know that the subgroup of S_4 which consists of all the even permutations of S_4 is the subgroup A_4. Therefore we can conclude that $A_4 \leq G_D$. As we see in Figure 7.8, there are five cubes that can be inscribed in the dodecahedron. Hence there are five distinct subgroups of G_D isomorphic to A_4.

Each of these subgroups contains four 3-cycles since A_4 contains four 3-cycles as we noticed in the tetrahedron. Therefore between the five subgroups in G_D, there are twenty 3-cycles in G_D. Since the elements of G_D permute the five cubes, then these twenty 3-cycles permute the five cubes.

Consequently, we have found twenty distinct 3-cycles permuting 5 elements. Such elements generate the group A_5 (Pap. 72). This implies A_5 is a subgroup of G_D, i.e. $A_5 \le G_D$. We know $|A_5| = 5!/2 = 60$ and we previously determined the order of $|G_D| = 60$. Therefore we must have $G_D \cong A_5$, confirming what we previously determined.

VIII. Historical Notes

The earliest signs of regular solids date back thousands of years to the Neolithic people of Scotland. Stone carvings of each of the regular solids were discovered dating back to this era and are on display in the Ashmolean Museum in Oxford (Weisstein "Platonic").

The regular solids were later found during antiquity. They were described by Plato in 350 B.C. in his work *Timaeus* where he related each of the five solids to an element of the universe: the tetrahedron to fire, the cube to earth, the octahedron with air, the icosahedron with water, and the dodecahedron to that which makes up the constellations and heavens (Weisstein "Platonic"). Euclid also discussed the regular solids in 300 B.C. in Books XIII – XV of his work *Elements* where he finds geometrical properties of the regular solids and discusses the concept of dual solids (Burn 60).

The fact that the rotational groups of symmetry of the regular solids are isomorphic to the groups A_4, S_4, and A_5 was discovered by F. Klein in 1874 and described in his work *Lectures on the Icosahedron* (Burn 60).

The regular solids appear in nature in crystallizations and many viral structures. For example, the herpes virus takes the form of an icosahedron (Wells) and the HIV virus is enclosed in an icosahedral capsid (Viral).

In everyday life the regular solids commonly appear in the form of dice, most commonly the cube, since their symmetry allow such dice to be "fair".

References

Armstrong, M.A. Groups and Symmetry. New York, NY: Springer-Verlang, 1988. pp. 2-5, 37-43.

Burn, R.P. Groups: a Path to Geometry. Cambridge, UK: Cambridge UP, 1985. pp. 57-61.

Papantonopoulou, Aigli. Algebra: Pure & Applied. Upper Saddle River, NJ: Prentice Hall, 2002. pp. 43, 72, 97, 135-139, 145, 466-468.

"Virus Structure." Microbiologybytes.Com. Oct. 19, 2004. Apr. 29, 2007. <http://www.microbiologybytes.com/virology/3035Structure.html>.

Weisstein, Eric W. "Archimedean Solid." From *MathWorld*--A Wolfram Web Resource. Apr. 6, 2007. <http://mathworld.wolfram.com/ArchimedeanSolid.html>.

Weisstein, Eric W. "Platonic Solid." From *MathWorld*--A Wolfram Web Resource. Apr. 6, 2007. <http://mathworld.wolfram.com/PlatonicSolid.html>.

Wells, Jason. "Herpes: the Virus, the Disease." Crispyneurons.Com. Apr. 13 2005. Apr. 29, 2007. <http://www.crispyneurons.com/index.php/Herpes:_The_Virus,_The_Disease>.

Images

Figure 1.1: "Platonic solid." The Five Convex Regular Polyhedra (Platonic solids). *Wikipedia, The Free Encyclopedia*. Wikimedia Foundation, Inc. Apr. 6, 2007. <http://en.wikipedia.org/w/index.php?title=Platonic_solid&oldid=120662789>.

Figure 5.1: Weisstein, Eric W. "Archimedean Solid." From *MathWorld*--A Wolfram Web Resource. Apr. 6, 2007. <http://mathworld.wolfram.com/ArchimedeanSolid.html>.

Figure 6.1: Cahir, Donal M. Duality. 1999. Relationships Between the Platonic Solids. Apr. 6, 2007. <http://www.ul.ie/~cahird/polyhedronmode/relation.htm>.

Figure 6.2: Weisstein, Eric W. "Platonic Solid." From *MathWorld*--A Wolfram Web Resource. Apr. 6, 2007. <http://mathworld.wolfram.com/PlatonicSolid.html>.

Figure 7.7: Jackson, Frank and Weisstein, Eric W. "Tetrahedron." From *MathWorld*--A Wolfram Web Resource. Apr. 6, 2007. <http://mathworld.wolfram.com/Tetrahedron.html>.

*Figure 7.8: "Dodecahedron." Regular Dodecahedron. *Wikipedia, The Free Encyclopedia*. Wikimedia Foundation, Inc. Apr 9, 2007. <http://en.wikipedia.org/w/index.php?title=Dodecahedron&oldid=120941112>.

*Figure 7.9 & 10: Lofting, C.J. The Dodecahedron. 2005. The I Ching, Small World Networks, and the Dodecahedron. Apr. 6, 2007. <http://members.iimetro.com.au/~lofting/myweb/The%20Dodecahedron.htm>.

All other images are original, drawn by author in Microsoft Word and/or Microsoft Paint.
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