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Special Aspects of Writing Mathematics Papers

Discrete Mathematics Version

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How to Read this Paper. This material is meant as a reference manual. Skim the whole thing now to get an idea what is in it. Then, as you are writing and need specific information, use the Table of Contents to locate what you need and read those parts carefully.

1. Introduction

Mathematics writing is different from ordinary writing and harder — in addition to all the requirement of ordinary good writing, there are additional constraints and conventions in mathematics. An additional constraint is that mathematics follows much more demanding rules of logic than ordinary discourse, and you must make your logic clear. Some of the additional conventions are those for defining new concepts and those for organizing the material through theorems and examples.

Although you have seen these constraints and conventions in the mathematics texts you have read over the years, you have probably not realized when and why they should be followed. They may seem highly technical and arbitrary, and thus not worth learning if mathematics is a side show for you. Why, for instance, should mathematical expressions be displayed in just certain ways, or bibliographies be organized is just certain ways? Indeed, as far as I can tell some mathematics conventions *are* arbitrary, for instance, the ones for bibliographies; I won't discuss them further or hold you to them when you write papers. But many of the conventions, including those for displaying mathematical expressions, are not arbitrary. There are good reasons behind them, and once you understand these reasons, you understand the nature of mathematics a little better and you become more perceptive about how to explain things in ways that you can carry over to your other writing.

The organization of this essay is simple. Each section discusses an aspect of mathematics writing my students have had trouble with, with examples taken from discrete mathematics. I try to tell you not only what you should do but also why. I have tried to put bigger issues, or issues you have to confront sooner, in earlier sections, but in some cases the order of sections is arbitrary.

Mathematics conventions are not rigid rules, and some authors write very well although (perhaps because) they break them. But before you can decide intelligently to disregard a convention, you have to know what the convention is, know why it is, and have some practice following it.

2. Getting Started

Start Early! Writing mathematics takes much longer than you think. Actually, all decent writing takes longer than you think, but mathematics is especially difficult and so especially time-consuming. There is even a joke about this, in the form of a riddle.

> Question: In what order do you write the chapters of a math book?
>
> Answer: 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, . . .

Think about this. Each time you draft another chapter, you find that *everything* that has gone before is not quite right to lead into your new chapter, and so you have to do all the earlier chapters over. Your earlier definitions, theorems and examples don't say quite the right thing in order to make the material in your new chapter fall into place.

The same joke applies to the sections of a paper. You'll see. Your ideas just won't come out correctly on the paper. They'll keep escaping from you.

3. What Kind of Mathematics Paper?

You will be writing *expository* mathematics papers. You will start with some issue (e.g., what is mathematical induction, or how can a complicated project be completed in minimum time), show that there is some definition, theorem, method, or algorithm that precisely and correctly resolves the issue, and give various examples. Most papers written by working mathematicians are *research* papers; these present new discoveries and are usually written in a terse definition-theorem-proof style with limited intervening commentary. Expository papers are more informal, with much more discussion, but they still should be carefully and visibly organized — carefully because that's the way mathematics is; visibly because mathematics is subtle, so readers need many guideposts to follow it. Ideas are made very precise and expressed in symbols as well as words. Key definitions, theorems, algorithms, examples and formulas are highlighted, numbered and referred to by number.

Not surprisingly, I think your text is a pretty good example of expository style. For other examples, I recommend certain expository mathematics journals in

Cornell Library: *The College Mathematics Journal, Mathematics Magazine* and *The American Mathematical Monthly.*

4. Set a Clear Objective

Generals do not like to be ordered into battle by political leaders unless they have a clear objective. If they don't have a well-defined objective, or have too many objectives, they are not likely to accomplish any of them, at great cost. It also helps if the objective is reasonable. The British in the Normandy landings had as their clear first-day objective the capture of the town Caen; this was not accomplished for three weeks.

Writing mathematics may not be quite the same as fighting wars, but the advice about objectives is right for both, for the same reasons. If you don't have a clear idea what you are trying to explain, or you are trying to explain too many things, you will, at great psychic cost, flail about and not explain anything well.

For your Math 9 papers, I have set the objectives for you, and given considerable detail about how to achieve them. These are reasonable objectives; I have some experience in such matters and these objectives have worked for previous classes. However, if you choose to write on a topic of your own, you will be relying on your own, less-developed sense of clear and reasonable objectives. Please check with me early if you pick your own topic, so I can tell you if I think you have bitten off too little or (more likely) too much.

You should always have the objective of conveying meaning, not just showing computations. This advice applies even if you are treating a computational topic. Suppose you write about using induction to prove summation formulas. You will need to display the details for several examples, but these details cannot carry the story themselves. You will need to emphasize for the reader the principles by which one determines what calculations to do, and you should summarize the method of calculation in words before or as you do the calculations.

5. Make the Material Your Own

To have a clear objective you have to have clear thoughts, and to have clear thought you have to make the material your own. This was the advice my Ph.D. adviser gave me. Start with what others have written down, read it carefully, work it over in your head, turn it around to your own way of thinking about it, until it is part of you. Then you will have clear and original ideas about it.

One way to make material your own is to keep working it over until you can explain it without referring to any text or notes. Think about it as you are falling asleep, or while you are taking a shower. Pretend you are explaining it to a friend at a blackboard. Take out a piece of paper and start drawing figures and writing formulas. Once you have gotten that familiar with your topic, you will find that you are doing it a little differently from how you learned it. Then you are ready to start writing.

Making a topic your own is valuable not just for writing. It's the best way to learn.

6. Know Your Reader

In *any* writing it is always important to ask yourself: For whom am I writing? In mathematics especially, the amount of explanation called for varies greatly with the reader's background. In this course, I will usually ask you to write for a student who has the same background as you, except that this student does not know the particular topic in your paper. (In particular, you may assume your reader understands our algorithmic language, even though it is not yet common for math books other than discrete math books to use such language.) Points that have been hard for you may be hard for your reader, and you should explain them carefully, using whatever approach finally worked for you.

Another important question is: How will your reader *use* your paper? Assume your reader really wants to learn the topic from you. Therefore, he or she will have to refer back frequently to earlier definitions, theorems, algorithms, explanations and examples — people have to go over mathematical items several times before they sink in. Thus, key elements of your paper should be marked so that they can be found easily. This requires highlighting and numbering, discussed later.

In any event, you are *not* writing for yourself. The twists and turns by which you finally came to understand your topic are not likely to fit in perfectly with your reader's mindframe; nor are they likely to be entertaining. It's usually best to go directly to the best explanation you have found.

7. Consistent Level of Detail

Once you decide the right level of detail for your reader, provide that level consistently. Consider the following lovely bad example from Price [1].

> Problem: Solve the equation $x^3 - 15x^3 - 28x - x = 85$.
>
> Solution: First combine the two terms in x:
>
> $$x^3 - 15x^2 - 29x = 85.$$
>
> Now subtract 85 from both sides.
>
> $$x^3 - 15x^2 - 29x - 85 = 0.$$
>
> Hence, $x = 17$.

This explanation is very uneven! The two easy steps are belabored, and then $x = 17$ is pulled out of a hat.

Sometimes it is appropriate for different sections of a paper to be written a different levels of detail. Typically, the last few sections are for advanced readers only.

8. The Shape of the Page

How things look on the page makes a difference in mathematics. Reading mathematics is hard and slow in any event, so a crowded page can be all the more discouraging and an "open" page, with wide-spaced displays,

wide interline spacing, clean type and eye-attracting marker items (examples, theorems, figures) can give the reader enticement to go on.

Most published mathematics is single spaced, but not all. *Please double space your paper.* This reduces the total amount on the page (accomplished in single-spaced books by having smaller page size than our standard $8 \times 11\frac{1}{2}$ paper) and makes each line easier to follow. It also allows for oversize math items within a line, which would otherwise require extra vertical spacing and would look funny. Also, when it comes time for you to make revisions, and for me to make comments, the spacing is needed for insertions.

If you are handwriting, wide interline spacing is even more important, as handwriting is harder on the eyes than printed symbols.

Wide margins, at least an inch, are important for the same reason as interline spacing: they limit the amount on the page and allow room for commenting.

Please number all your pages. This is especially helpful if you should staple them out of order (which happens) or if I get them out of order while reading (this also happens).

9. The Shape of an Argument

There are many ways that format (shape) can affect the ease with which readers will follow your reasoning (argument). I have already mentioned how key items should be numbered and highlighted, for easy referencing and to indicate their importance. In the next few sections I discuss several other ways in which structure can help comprehension: paragraph division; sentence structure, use of connective words, and displays.

10. Paragraphs

Writing isn't formatted into paragraphs to look nicer on the page or to indicate where you took a deep breath. Rather, paragraphs should separate the development of one idea from the next.

Specifically, a paragraph in a proof should represent a major segment of a proof. For instance, if a proof of $A \Longleftrightarrow B$ is short, then one paragraph should prove that $A \Longrightarrow B$ and the other should prove $B \Longrightarrow A$. If, however, $A \Longrightarrow B$ is long, then it too should be broken in parts. For instance, if you are presenting a long proof that

$$G \text{ has an Euler cycle} \implies [G \text{ isconnected and has even degree}],$$

then devote one paragraph to

$$G \text{ has an Euler cycle} \implies G \text{ is connected},$$

and another to

$$G \text{ has an Euler cycle} \implies G \text{ has even degree.}$$

In student papers (and alas, sometimes in published papers), I often find paragraphs that should be broken up, or situations where the last sentence of one paragraph belongs at the start of the next. Look for such things in your work when you revise, and correct them.

11. Sentences

Mathematics consists of thoughts, and thoughts are communicated in sentences. Therefore, *Mathematics is written in sentences.* This doesn't mean you use words only. Symbols, equations and displays are important parts of those sentences; each equation is a clause. But one unpunctuated display after another, with no connecting words, do not make sentences.

In general, a sentence should make just one point. Thus, in a proof, most sentences should give just one step of the deduction. Having many steps jammed in the same sentence makes difficult reading.

Also hard to read is having one step spread out over several sentences. Specifically, *don't state a result in one sentence and, without forewarning, only give the reasons in later sentences.*

Example 1. Don't write

> Thus the number of people at a party who shake hands an odd number of times is even. Too see this, let the people form the vertices of a graph, and the handshakes the edges. Since the sum of the degrees of any graph is even, no graph can have an odd number of vertices of even degree.

Example 2. Even harder for the reader is when the offending sentence ends in a display:

> The number of people at a party who shake hands an odd number of times is even, because
>
> $$\sum_{v \in V(G)} \delta(v) = 2|E|. \tag{1}$$
>
> Eq. (1) is true for the following reasons. . . .

If this display happened to come at the end of the page, then the sentence explaining it would be out of sight on the next page. That would be even worse.

Why are the situations illustrated above bad? Mathematics is difficult, and readers expect to have to stop sometimes and do a little thinking before believing what they have just read. Every period is a marker at which an experienced reader knows s/he may have to stop. (This is different from most reading – one does not expect to stop and think at periods.) However, not every sentence requires a stop. The claims in some sentences have been proved earlier. The claims in some sentences are obvious if you have followed what has come before. But what if you read a sentence and its claim is not obvious to you, and the author gives no hint about how hard the proof is supposed to be? Then you have to start trying to prove it for yourself. If you can't get it, finally give up, and only then discover that the required steps begin with the next sentence, you have suffered frustration and wasted time.

The point it: it is incumbent on a mathematical writer to always tell the reader the status of a claim, and to announce that status within or before the very sentence where the claim is made. Was the claim proved previously? Will it be proved in the next few sentences? Is it a hard theorem that the reader is being asked to accept without proof? Is it obvious? Is it a claim the reader should be able to see with a moment's thought?

Let's see how the two examples above fail to guide the reader properly. In Example 1, the justification for the first sentence comes in the later sentences, but there is no warning that this will happen. Indeed, the word "Thus" at the start of the first sentence suggests that the claim follows from some previous sentences. So the good reader, instead of going on to the second sentence, will mistakenly look back at previous sentences. Deleting the word "Thus" would help; there would be no clue as to where the justification is instead of a misleading clue. However, the usual understanding is: if there is no clue, then the claim should be easy to show based on what has *come before*.

Finally, the last sentence of Example 1 is also problematic. The word "Since" says that the first clause justifies the second. Is this obvious, or something that the reader should be able to see with a moment's thought. If so, the sentence is ok, although it would be better if the second clause began with "it is easy to see that". If the argument is not so easy, then more explanation is needed.

In Example 2, the first sentence says that the claim in its first clause is explained by Eq. (1). Now, if Eq. (1) had been proved earlier (or is something the reader could be expected to know, say, because it was in the textbook for our course), and the author had reminded us of that fact, everything would be fine. But apparently it had not been proved earlier, because the author proceeds to prove it. But s/he does not tell us s/he is about to prove it. And since Eq. (1) is a display, our eyes do not automatically see the start of the next sentence as we read the display. We certainly don't see the next sentence if the display ends a page.

Fortunately, it is very easy to avoid misleading the conscientious reader in these ways. All you need is to make regular use of some short phrases at the beginning of the sentence containing the claim.

If the claim follows from something that was just said, begin with

thus, or *so* (See further discussion below.)

If the claim was shown earlier, or follows directly from something said earlier, say:

As we showed before, or

See Eq. (6) (a previous equation)

It follows from Theorem 3

From Euler's Theorem (if the claim follows from a well known result)

If the proof is about to follow, say:

As we will now show

If the proof will not be given and the reader should just accept the result, say:

it can be shown that, or

it is not hard to show that (if in fact the proof is not hard but the reader shouldn't bother to try to figure it out)

it is a deep theorem that (if the proof is very hard)

If the proof should be easy to figure out, say:

It is easy to see that

Remember, punctuation at the end of a display is particularly important. A display should not end without punctuation. If the thought has ended, and I have been told all I need in order to know why the display is true, it should end in a period. If more justification is to come, the display should end in a comma or semicolon.

12. Connective Words

The logical connection of consecutive sentence should always be made clear. Connective words and phrases – such as *therefore, thus, so, however, similarly, it follows that, nonetheless, moreover* – are very important and should appear frequently.

"Thus", "so", and "therefore" at the start of a sentence are particularly important and very specific. They mean that the sentence or clause they begin is a logical consequence of the previous sentence or clause. Therefore, if the next sentence would still make sense and be true even if you had not included the previous sentence, then the next sentence may *not* begin with "thus" or "so". Example:

> Let the vertices of graph G be the students in Math 9, and let there be an edge between two vertices if the two students are friends. Thus the number of vertices of odd degree is even.

Here the use of "thus" is incorrect, because the theorem that the number of vertices of odd degree is even does not depend on what the vertices and edges represent. However, if you replace the second sentence with

> Thus the number of students in Math 9 who have an odd number of friends in the course is even,

then the use of "thus" is legitimate (why?), but you probably should give more explanation.

13. Displays

Any long expression with mathematical symbols is **displayed** — placed on a line by itself (and usually centered), with extra vertical space around it. Expressions that are not displayed are said to be **in-line**.

There are three reasons for displays. First, if an expression is particularly important, you draw attention to it by displaying it. Second, longer mathematical expressions tend to be tall. In single-spaced math, to fit something like $\displaystyle\sum_{k=0}^{5} ar^k = \frac{a(r^{n+1}-1)}{r-1}$ in-line requires adding disconcerting interline space (see it?), or else making the symbols disconcertingly small, like this: $\sum_{k=0}^{5} ar^k = \frac{a(r^{n+1}-1)}{r-1}$. Third, if you put the expression in-line, it might come at the end of a line and have to be broken across two lines. You can hyphenate words, but mathematical expressions are usually longer than words and don't bear line breaks well. (You certainly can't hyphenate them; the hyphen would be mistaken for a minus sign.) So if an expression would have to be broken if it appeared in-line, it should instead be put in a display. (If you must put an expression in-line, and must break it between lines, it is best to break it right after an equal sign or other main verb.)

A display should be numbered if you find that you refer to it anywhere other than within a few lines of it. The numbering can appear either on the left, as in

$$(2) \qquad \sum_{k=0}^{n} k = \tfrac{1}{2}\, n(n+1),$$

or on the right, as in

$$\sum_{k=0}^{n} k = \tfrac{1}{2}\, n(n+1), \qquad (3)$$

but be consistent. Just below a display you can refer to it as the "previous display", but two pages later it's too long-winded to refer to "the third display on page 7". Besides, if you modify your paper, that display probably won't be the third on page 7 anymore, so you will have to rewrite such references with every revision. That's why we use numbering.

14. Presenting Arguments through Displays

The fact that calculations *can* be written as one long display without words doesn't mean that they *should* be written this way. Mathematics does not consist of calculations alone! Very elementary algebra need not be explained, but more complicated calculations should be, especially if the calculations are justified by mathematics that the reader is just learning. There are two ways to provide the explanation. One way is to put it before, between and after the lines of calculation. For instance, suppose you are proving that $(1+x)^n > 1+nx$ when $n > 1$ and $x > 0$. Suppose you have already verified the basis case ($n = 2$). You could proceed this way:

> For the inductive step, assume we have already shown case $n-1$:
>
> $$(1+x)^{n-1} > 1 + (n-1)x.$$
>
> Now,
>
> $$(1+x)^n = (1+x)^{n-1}(1+x).$$
>
> Substituting the assumption, and noting that $(1+x) > 0$, we obtain
>
> $$(1+x)^{n-1}(1+x) > (1+(n-1)x)(1+x).$$
>
> Expanding the right hand side gives $1 + nx + (n-1)x^2$ and
>
> $$1 + nx + (n-1)x^2 > 1 + nx,$$
>
> since $x^2 > 0$ and $n-1 > 0$. Combining all these steps, we get
>
> $$(1+x)^n > 1 + nx,$$
>
> which is case n.

Notice I didn't define or use $P(n)$; you don't *have* to use that notation.

An alternative approach is to put all the commentary on the right. To make the commentary fit, you do need abbreviations like $P(n)$ for the nth proposition. So assume you have already given the basis case and defined $P(n)$. You could proceed as follows:

Inductive step, $P(n{-}1) \Longrightarrow P(n)$:

$$\begin{aligned}(1+x)^n &= (1+x)^{n-1}(1+x) \\ &> (1+(n{-}1)x)(1+x) && [P(n{-}1) \text{ and } 1+x>0] \\ &= 1+(n{-}1)x+x+(n{-}1)x^2 = 1+nx+(n{-}1)x^2 && [\text{algebra}] \\ &> 1+nx. && [x^2>0,\ n{-}1>0]\end{aligned}$$

Thus $P(n{-}1)$ implies $[(1+x)^n > 1+nx]$, which is $P(n)$.

Finally, as noted in class and the text, you can put reasons over equality and inequality signs. For instance, in the second line above you can put $P(n{-}1)$ over the inequality.

15. Titles

Every paper should have one. It should be informative without being too long. Choosing a good title in a mathematics paper is not so easy. Often a paper hinges on a concept that is defined only within the paper itself, so using the name of that concept in the title may convey no meaning at all. Since the goal of your main paper in Math 9 will be to solve a real-world problem using discrete mathematics, its title should suggest your problem and indicate that math is used to solve it.

16. Introductions

Again, every paper should have one. In short papers (such as your induction paper), it need only be a brief paragraph. In longer papers, it might be a whole section. Mathematics papers are hard to read, and for encouragement your reader deserves to be informed of what she is getting into and why she should care. Giving a good introduction is difficult in the same way giving a good title is: a satisfactory explanation may hinge on concepts you haven't introduced yet. But in a paragraph you can give a rough idea of the key concepts (admit that it's rough) and what you will do with them.

17. Division into Sections

Break papers of longer than 4 or 5 pages into sections. Your main paper will have longer and fewer sections than this essay, because your paper will have a single topic. In a paper with only a few sections, the introduction should probably include a brief description of what each section will cover. At the start of each section remind the reader what the section is about. How does it fit into your scheme of things? Why is it there at all? Like the introductory section, these reminders will give the reader encouragement and guideposts. They will also help you. Writing them will force you to think hard about (and then maybe revise!) what you are trying to do and the order in which you are trying to do it.

18. Definitions

Definitions are a major reason why mathematics writing differs from general writing. Most disciplines don't need to make definitions explicit nearly so often as mathematics does — they don't need to be so precise nor do they deal so regularly with situations outside common experience. Mathematical writing involves defining both words/phrases (e.g., graph, floor function) and notation ($\lfloor x \rfloor$, $u \sim v$). Notation is important because, if you use a new concept frequently, you need a shorthand way to refer to it or you will tie yourself in verbal knots.

When you give a definition, you can do it "in-line" (within a paragraph), but the word or phrase being defined should be highlighted, by underlining, by *italic print*, or by **boldface** (but not by quotes; see below). Boldface is now the most common format, since underlining is not common in typeset material and italic already has a use in both mathematical and ordinary writing to indicate *emphasis*. Here's an example of a definition: "A **prime number** is a positive integer with no positive integer divisors other than 1 and itself." Another format is to display definitions just like theorems are displayed. The display format should be reserved for the most important definitions. For instance, in the definition of graph might be displayed, but the definition of a complete graph (if you need it at all) can be done in-line.

Quotes aren't used for definitions because they are used instead to indicate that a word is being given a somewhat unusual meaning. For instance, consider the sentence

To proceed, "simplify" $\sum_{k=1}^{n} k^2$ to $\left(\sum_{k=1}^{n-1} k^2\right) + n^2$.

Here the quotes are appropriate because one wouldn't usually call this manipulation a simplification.

There are two types of definitions, **global** and **local**. Global definitions apply throughout your paper. Most global definitions introduce words or notation. In contrast, local definitions apply only briefly, say, to the current example or current paragraph. Usually, local definitions are assignments of values. For instance, "Let $h_n = 2^n - 1$" assigns a specific sequence to the generic sequence $h_1, h_2, h_3, \ldots$. A few paragraphs later it is acceptable to say "Now let $h_n = 3^n + n^2 - n$." Generally, local definitions don't require special highlighting. But note: The item being defined should always come on the left-hand side of the equation. Thus, "Let $2^n - 1 = h_n$" is *wrong*. Why? Because the use of an equal sign with "let" is special; the quantity on the right is being assigned to the quantity on the left. But the algebraic meaning of $2^n - 1$ is fixed; you can't "let" it be some arbitrary thing h_n.

What to Define and When. Under what circumstances should you create a new definition, or state an old one? And where should you put these definitions in your paper?

You probably won't have to invent any new global definitions. You are writing an expository paper and the definitions in the literature are time-tested. (In research papers mathematicians regularly invent new global definitions. Indeed, sometimes a new definition is the most important thing in a paper, because it gives people a new way of thinking about something.) For an expository paper the issue is: which old definitions should you state? Answer: it depends on your reader. In your second paper I will probably ask you to write about graph theory assuming your reader has never seen graph theory. Therefore, you will have to define each graph theory concept you really use.

Actually, there is one circumstance under which you might need a new global definition — if you find you keep needing a concept for which there is no standard simple phrase. For instance, suppose you repeatedly need to say that one graph has fewer edges than another. It gets tedious, and takes space, to keep saying "G has fewer edges than H." So make a definition. For instance, write "We say graph G is **smaller** than graph H if G has fewer edges." A formal definition is necessary, because "smaller" is not already a defined term in graph theory, and without a clear announcement your reader won't know if you mean fewer edges, or fewer vertices, or drawn smaller on the page, or what. The opportunity to override the ambiguity of words in ordinary English by defining them narrowly is one of the most powerful aspects of mathematical writing.

Now, *where* should you put your global definitions? For the most part, you will be tempted to define a word immediately after you first used it. This shows lack of planning and lack of thinking about what will read best. When this policy leads to definitions being parceled out in twos and threes in the early part of the paper, it works fine. But if it means you interrupt a difficult, key proof late in the paper to give two more definitions, it is disruptive and confusing. Before you start writing a math paper, you should make a list of all the definitions you think you need, and then try to see to it that they all get introduced relatively early. (Of course, you will not yet realize exactly which definitions you will need, so after the first draft you will have to go back and add more definitions and delete others.) The extreme version of this approach is to put all the definitions in a definitions section near the beginning. (Our text does this in Ch 3.) This isn't always best either: a long section of definitions can be deadly boring.

As for local definitions, where is easy: put them at the start of the material that uses them. But when should you create a local definition? Answer, when it would be confusing or require verbal contortions to avoid it. For instance, in proving Euler's Theorem, suppose you want to say something like:

> We find a cycle in our graph. If the cycle contains all edges, we are done. Otherwise, remove the cycle and look for another cycle in the remaining graph. If the cycle we find does not contain all the edges, then…

This is getting unclear; for instance, in the last sentence, *which* cycle are you referring to and from *which* graph should it contain all the edges? How much better to write:

> We find a cycle C in our graph G. If $E(C) = E(G)$, we are done. Otherwise, find a cycle C' in the graph $H = G{-}C$. If $E(C') \neq E(H)$, then…

Admittedly, this is a bit of overkill. The second sentence could be "If C contains all the edges of G, we are done." Indeed, if you haven't introduced $E(G)$ previously to mean the edge set of G, the less symbolic version is probably better. Nonetheless, if you need to refer to an item more than once, it is probably best to give it a name.

Dangers to Avoid. There is a side effect to making definitions that you have to watch for. If a word with a ordinary English meaning is given a specific mathematical meaning by a definition, then its looser ordinary English meaning is no longer available for you. The word has become a **reserved word**. For instance, to say that two vertices are "connected" has a specific, narrow meaning in graph theory — they are joined by a path. Therefore, the common English meaning "immediately attached" is no longer available. If you want to say that two vertices have an edge between them, you cannot say they are connected. (What should you say instead?) This is true even if you haven't given the mathematical definition of connected in your paper. That definition is so common in graph theory that you must honor it even if you don't use it. To do otherwise will simply confuse some readers.

Similarly, if you define "smaller" for graphs as on the previous page then you can no longer use smaller in connection with graphs with any other meaning.

Here are some other defined terms that are so standard in discrete mathematics that you cannot use them in their ordinary English sense even if you have not stated their defined meanings:

induction   combination   series   loop

Another thing you shouldn't do, once a word w has been given a mathematical definition, it to use an ordinary English synonym w' as if it has the same mathematical meaning as w. For instance, once "connected" is defined, don't occasionally substitute "attached" (unless you have stated explicitly that "attached" is defined to mean "connected"). An experienced mathematical reader, upon seeing "attached" in your paper

after you have defined "connected", will assume you are using both words because you need two different refinements of the ordinary English concept of bound together. (Maybe you are using "attached" in the way other authors use "adjacent".) Such a reader will then attempt to find where you defined attached and will get frustrated.

If two ordinary English synonyms have been given different mathematical definitions, it's even more important not to use them as synonyms. Don't use "connected" when you mean "adjacent". Don't use "equivalent" when you mean "equal".

19. Hypothesis and Conjecture

"Hypothesis" is a reserved word that is used differently by mathematicians than by other scientists. To most scientists, a hypothesis is a statement that is believed but has not been confirmed. Mathematicians call such a statement a *conjecture*. In mathematics, a hypothesis is merely the if part of any if-then statement, and is distinguished from the *conclusion*, or then part.

For example, the Four Color Theorem says: If a map is planar, then it can be drawn with at most 4 colors. In the almost 100 years between the time this claim was first proposed and the time it was proved (1976), it was known as the Four Color Conjecture, not the Four Color Hypothesis. The theorem does involve a hypothesis, namely that we are talking about planar maps (as opposed say, to maps drawn on a doughnut, which can require as many as 7 colors).

When stating theorems, students often omit the hypothesis. For instance, ask a student what the Pythagorean Theorem is, and s/he might say "$a^2 + b^2 = c^2$". But that's just the conclusion. The hypothesis is that a and b are the lengths of the sides of a right triangle, and c is the length of the hypotenuse. You can't prove a theorem if you don't appreciate what is being assumed, so math teachers are always asking students "What's the hypothesis?"

20. Examples

Examples really help to make abstract concepts clear, so a good expository paper contains many — more examples than definitions and theorems. Examples are like definitions, in that they can appear in-line or be highlighted by indentation and extra spacing. For a very brief example, the in-line method is fine. For instance, after defining "prime number", you could write: "For example, 5 is prime and 6 is not." However, a lengthier example should be displayed and numbered, especially if it is a key item of your work (as in your first paper). Not only does this format draw attention to your example, but is also makes later reference easy (e.g., you can say "see Example 3"). If an example is a sample problem, make clear where the solution begins and ends.

21. Figures

Figures can be extremely helpful in an expository paper, just as they are in books. Each figure should be numbered (for easy reference later) and inserted shortly after the first reference to it. (Another convention is to put all figures at the end. If you use this alternative, please say so the first time you reference a figure.) Usually, each figure should have a caption as well; e.g., "The graph of a tournament with 5 teams". If you know how to use computer software to produce a good figure and place it into your paper, great. (I'll even demo some programs that produce discrete math figures). However, producing good technical figures takes time, and now is probably not the time to learn how. It's fine if you insert your figures by hand. You'll probably need more space than you first think.

22. Movies and Hypermedia

Actually, figures are old hat. Now, if you submit your paper electronically, it is possible to include animations. For instance, you could create a movie using Mathematica, save it as a QuickTime file, and then paste this file into your paper using a wordprocessor that recognizes QuickTime. Movies can be very effective in parts of discrete mathematics, for instance, in illustrating algorithms on graphs.

In fact, movies are just one of the many "hypermedia" capabilities that are becoming available. Another is sound. Yet another is the ability to "link" any parts of your document, to other parts of your document or to other documents around the world. When you click on a link (say, a word highlighted a certain way), you automatically switch to the other end of the link. It becomes much easier to jump around while reading a document. This is sometimes an effective way to learn mathematics, even though formally mathematics is quite linear.

For now I just want you to be aware of these capabilities. I do not recommend going out of your way to acquire the requisite software, or to learn how to use it well. But it is conceivable that in just a few years professional papers, and even short student papers, will routinely be hypermedia documents.

23. Theorems

Any theorem in your paper should be highlighted — I suggest indenting it or putting extra vertical space around it. If you have more than one theorem, number them. For instance,

> **Theorem 2.** In any graph, the sum of the degrees of the vertices is twice the number of edges. In symbols,
>
> $$\sum_{v \in V_G} \delta_v = 2\,|E_G|.$$

Other formats are allowed; browse and see. Why highlight and number theorems (and other key items such as examples)? Because mathematical arguments can be so complex. Formal statements of theorems (and other key items) serve as touchstones, like having an outline within the text body. Highlighting also makes it easy for the reader to refer back to key specifics. Also, in a paper with many results, calling some theorems, some lemmas, some propositions, etc, helps to make the relative importance clear. Similarly, explicitly marking the beginning and end of proofs helps the reader know where s/he is.

It's best to state a theorem before proving it. Math arguments are long and complex, hence hard to follow even for smart people; thus it is unfair to expect someone to commit him/herself to the effort to follow an argument without knowing where it's supposed to go.

Whereas a theorem statement should precede a proof, the proof need not follow immediately. If the main purpose of your paper is to show an application of the theorem, then you may defer the proof till later, or sometimes not give it at all. In other words, not everything in a math paper need appear in logical order. But when you go out of order, you must immediately inform your reader of this fact. For instance, immediately after the statement of a theorem, you might write "We defer the proof until after we present the main application." If you don't include the proof of a theorem, be honest about the reason. If you don't understand the proof, say so. Normally, you would include the proof of your main theorem, but if the proof is very lengthy, or very complicated, or you don't understand it, you might not. It is quite common not to include the proof of a peripheral theorem. For instance, if there a theorem that solves a similar problem, or a more general problem, you might want to mention this problem and its theorem, but it's quite reasonable not to prove that theorem.

What if your paper is devoted to a specific example where, once a mathematical representation is found, the answer is obvious? For instance, suppose you wish to find the minimum completion time for a certain project where, once it is represented as a graph, you can eyeball the answer. Then you should point out that, had the example been more complicated, a general method for dealing with the mathematical representation would be needed, and you should describe and justify a general method. Without such general arguments, you don't have a math paper.

24. Levels of Confirmation (prove, verify, show, illustrate)

In ordinary discourse, to prove something means to give any sort of fairly convincing evidence. In mathematics, "prove" has a much stricter meaning; you have proved something only if you have given a airtight argument — airtight because it harks back to definitions or to other theorems that hark back to definitions. Lesser levels of confirmation are often quite useful, but they are referred to with different words such as "show" and "illustrate".

For instance, consider the formula for the sum of a finite arithmetic series:

$$\sum_{k=1}^{n}[a+(k-1)d] = an + \tfrac{1}{2}n(n-1)d. \tag{4}$$

If you state this theorem and then say "For instance,

$$3+5+7+9 = 24 = (3)(4) + \tfrac{1}{2}(4)(3)(2)\text{''}, \tag{5}$$

you have merely *illustrated* how to *use* the theorem (with the case $a = 3$, $d = 2$, $n = 4$). If you actually prove the case

$$\sum_{k=1}^{n} k = 1n + \tfrac{1}{2}n(n-1)(1) = \tfrac{1}{2}n(n+1),$$

you still haven't proved Eq. (4), though you may have indicated how the general proof would go.

Mathematicians use "verify" somewhat like "prove". If you say you are going to verify a theorem, then you must prove it. But you can also verify a numerical claim; this is a much less demanding type of verification. For instance, to check that Eq. (4) is correct in the special case of Eq. (5) is merely a verification.

Also, you cannot prove, or verify, an example. You cannot prove a problem. Such statements are examples of **type errors**. Only *statements* can be proved, and problems and examples are not statements.

Can you prove an algorithm? People do use that phrase — I may even have said it. But what they really mean is that they will prove that the algorithm is correct.

In summary: "prove" has a very strict meaning; "show" is looser, and "illustrate" is looser still and refers to examples. Although illustration is the loosest, it is very important. Sometimes good examples will do more to help the reader understand and believe a result than a complete proof will.

25. Algorithms

When an algorithm is a main part of your paper — when you need to explain the algorithm carefully, or prove that it works, or analyze its complexity — then, like key theorems and examples, the algorithm should be displayed. Moreover, it should be written in algorithmic language. This language limits ambiguity and makes it possible to point to specific aspects of the algorithm by referring to specific lines or variables. Our text provides many good examples.

26. When to Give Credit

The conventions for giving credit to others are somewhat different in mathematics than in general academic writing. Briefly put, direct quotes and almost direct quotes must be credited, but paraphrase generally need not be, and mathematics papers rarely include direct quotes. Thus, while a humanities paper is full of footnotes giving credit, in a mathematics paper the credits are less frequent (and they typically appear in text). The only way that people must be credited is for recent theorems they create. This is all elaborated below. Pay close attention, since failure to give proper credit is plagiarism, a major academic sin.

Rightly or wrongly, mathematics is regarded as having an existence independent of the words used to describe it. Thus the fact that Maurer & Ralston describe mathematical induction in different words than other authors doesn't give us any special credit. If you use induction, and you learned about it from our text, you should *not* reference us when you start discussing induction, and you certainly should not quote verbatim our description of induction (as if it were gospel). Similarly, if you define graph, don't make it sound like we invented the definition.

On the other hand, since theorems are considered the big thing in mathematics (even though definitions are often as important), you must give Smith credit if you state or use a theorem that was first proved by Smith. If the result is a recent one, you should reference Smith's paper. However, if the result is classical, e.g., Euler's graph theorem, it suffices to call it Euler's Theorem; no reference to a source is necessary. (In mathematics, a result is "classical" if it is a well known and already appears in many texts; occasionally results proved as recently as 20 years ago have already become classical.)

Should examples be credited? Again, the answer is "usually no" (even though a good example can be just as important and just as original as a theorem). If you use the exact words or even just the exact numbers of an example from some book, then (and only then) give credit; but in general you shouldn't be using problems verbatim anyway. Why? Because in order to write about an example with understanding you should internalize it and make it your own before using it. And if you have internalized it, you can recreate it, no doubt with different numbers.

Should algorithms be credited? If you copy an algorithm word for word, then you have to reference where you got it. Also, if it is a recent creation of the author's, again you have to give him/her credit. But if its an algorithm in an undergraduate text, you can assume it's part of the public domain (unless the text says otherwise). Even if the algorithm has a name attached (e.g., Warshall's Algorithm), you don't have to give a reference to the original paper unless your text does. Warshall's algorithm is as much part of the public domain now as Euler's theorem, or Taylor polynomials in calculus.

If the algorithm is part of the public domain, why not study the algorithm until you have internalized what it says, and then produce your own version. If you do this, you need not reference the source, even if your wording turns out to be almost the same. The same goes for using exercises from a text. More important from your point of view, by internalizing the algorithm or exercise, you have learned something.

If your whole line of thinking comes from one source, it *is* appropriate to give *one* reference to it. Somewhere near the beginning of your paper, say something like "In writing this paper, I have drawn heavily on Bogart [1]" (where Bogart is reference 1 in your bibliography) or "Much of my material is adapted from lectures by Smith [2]".

27. Footnotes and References

Footnotes are rarely used in mathematics papers, though they are making a bit of a comeback. First, marking them with superscript numbers might be confused with exponents. Second, it used to be (before math was composited by computer) that mathematical symbols weren't always available in the smaller size type used for footnotes.

So what do you do? If you need to refer to an outside source in the text, write $[x]$ (where x is the number of the reference in the reference list at the end). For instance, if Maurer-

Ralston is reference #1, you might write: "The following application is treated in [1]", or "The following is treated in Maurer & Ralston [1]". If you want to give a specific location in the reference, a Theorem or Example number is at least as valuable as a page number: "The following application is treated in [1, Ex. 3.5.2, p. 146].

(What about footnotes that contain peripheral remarks instead of references? These too should be avoided; put the material in the text. One way to mark such material as peripheral is to put it in a separate paragraph in parentheses, like this paragraph!)

If you don't consult any books or papers, then it's fine to have no reference section. If you do consult any, then you should reference them, even if you ended up using nothing from them directly.

28. Writing Mathematics on a Computer

Publishers call mathematics penalty copy because it is so hard to "set" on the page. Mathematics involves (among other things) subscripts and superscripts, numerous styles of type, special and/or large symbols ($\sum$, $\in$, $\subset$), and expressions that must be stacked vertically (fractions, limits under and over summation signs). Traditional printing involved setting type across simple lines, and many mathematical features interrupt that flow.

I assume you are used to using computers for general wordprocessing. The versatility of computer writing, especially for revisions, is so great that I cannot conceive of going back to previous methods. My documents would be shorter and much less thoroughly revised. However, most word-processing programs are set up to go straight across lines also, so for writing mathematics they leave you with the same problem traditional publishers had.

There are special computer programs for writing mathematics. The Cadillac of these (because of its versatility, its power, and its "platform-independence") is TeX (and its extensions, such as LaTeX ad AMS-TeX); most scientific publishers use it now and most professional mathematicians have learned at least the basics. (This memo is written with TeX, and so is our textbook.) Various versions, that run on various computers, are even available free at Swarthmore. However, TeX is *very* hard to learn. It doesn't work like most Macintosh word-processing programs; for instance, you can't just click on a spot on your screen and start putting math there.

Today, the most powerful of general purpose word-processing programs also have pretty good mathematics capabilities, and sometimes these are not hard to learn. For instance, Microsoft Word 5.0 has a module called Equation Editor, which gives you "templates" for creating mathematical expressions as pictures which you paste into your main document. In about two hours, with a manual, I learned how to use it pretty well, and it can do most of the things I want.

However, I will assume that most of you do *not* have a word-processor with powerful math capabilities, or don't know how to use those capabilities. What should you do?

Answer: Do by computer only those math expressions that are easy to do with the writing programs you know (see below). For more complicated things, you shouldn't spend more than an hour trying to do them. (But use the hour productively – get a friend or a consultant, or me, to help you; use the on-line help or find a manual.) Anything you can't get to look good enough on the screen you should write in by hand. At first, you will probably grossly underestimate the amount of space you need for hand insertion. But if the rest of your paper is written on a computer, it will be easy to output another copy with more space. Just don't forget to write in everything you left out; as you do revisions this will be a bit of a pain.

(Exception: If you expect to write a lot of math papers in your life, then maybe this is the time to start to learn how. Come and see me to discuss this how to start.)

Some aspects of setting mathematics are easy to master, and others can be avoided. Almost all word-processing software can create subscripts and superscripts easily, and you should use them. The next two subsections tell you how to deal with two major "problems" in setting mathematics: styles of type and fractions.

Fonts for Math. A **font** is a style of type, e.g., roman or italic. In books and journals, letters representing mathematical quantities, such as a, y and $f(x)$, are mostly set in italic, as just done, to distinguish them from ordinary roman text. (Some mathematics is set in other fonts, e.g., Greek letters for angles in trigonometry, boldface for vectors in linear algebra, and symbol fonts for special math symbols like $\cup$ and $\in$.) Math italic is different from ordinary italic, because only the letters are slanted, not the numerals, not the operators (like +), and not the punctuation (parentheses, brackets, exclamation points).

Why special fonts for math? One reason is that, without them, sometimes it would not be clear what is intended, a mathematical expression or an ordinary word. For instance, consider the sentence, which actually appeared in a paper written for Math 9 in 1991: From $P(n)$ we know that an is an integer. What the writer meant was: From $P(n)$ we know that an is an integer. Only the italics makes it certain that, in the first "an", we mean the quantity a times the integer n. (Notice, too, that without italics "a" could be confused with the English word "a".) Another reason for special fonts is: if you have several quantities, and each has an English name beginning with "M", you might want to represent each of them mathematically by the single letter m. Then you need to use a different font for each meaning.

What should you do about fonts. *Do not attempt to create italic math using a regular (text) italic font.* To keep the numbers, parentheses and symbols from being italic you would have to go back and forth between roman and italic so often it would drive you crazy. Unless you have a "math italic" font, it's best to avoid italics entirely and use an alternative convention stemming from the days of typewriters without interchangeable type balls: Use a single font and put extra blank space around mathematical letters and expressions. For instance,

Consider the variables x and y in the equation y = 2x+3 for a line.

If you do use just one font, stay away from san serif fonts (no squiggles on letters). In such fonts 1, l and | look alike, as do 0 and O and some other pairs.

For special math symbols, you will probably find most that you need in the "option keyboards" for your regular word-processing fonts. Try holding down one of the special keys (option, cloverleaf, control) on your Mac while you hit letter keys. To see in advance what you will get, open up the KeyCaps desk accessory that is probably listed under the Apple Menu (top left) on your Mac screen. Caution: don't use any of these special symbols when sending email. They probably won't transmit correctly. They should only be used within a word-processing document

Whatever font you use for a mathematical symbol, *make sure you use that font all the time.* For instance, if you are talking about team t in a tournament, do not use t sometimes and t other times. Why? Mathematics is font-sensitive. Trained readers expect the same letter in different fonts (or with different accent symbols) to mean different things. For instance, a statistics paper might well use x for a data value, $\mathbf{x}$ for a sequence of data values, and $\bar{x}$ for the average value of the data.

Mathematics is case-sensitive too, that is, upper and lower case mean different things. For instance, that statistics paper probably uses X for the "random variable" of which x is a value. So, what do you do if you want to start a sentence with a variable you have named a? Answer: Don't write A. Start with lower case, or (better) rewrite the sentence. For instance, instead of starting "$a > 0$ because", start "Quantity a is positive because".

Warning 1: Some computer software for mathematics is *not* case-sensitive. For instance, most versions of BASIC will treat the variables B and b as the same.

Warning 2: If you do use some italic letters, then what you see on your screen is probably *not* exactly what will print out. On the screen an italic letter will appear to overlap the roman symbol to its right, but in the printed version this will go away. This overlap on the screen is disconcerting, but your best solution is to ignore it. (If overlaps persist on printed copies, see me, or stop using italics! I've never had spacing trouble with italics, but I have had trouble on Macs with printing boldface.)

Fractions. Fractions in "built-up" form like

$$\frac{y_2 - y_1}{x_2 - x_1}$$

are especially hard to produce without special software. (Fortunately, they don't occur very often in discrete math.) So what should you do about fractions? Within a paragraph, for simple fractions you can avoid the problem by using "shilling" style, that is, a/b instead of $\frac{a}{b}$. But as soon as the numerator or denominator involves sums and differences, the shilling approach requires parentheses and becomes hard to read. When guilt-up form looks much better, you should use it, and you should probably write it in by hand.

29. Antecedents

If I write "Antecedent?" on your paper (or just "ante?"), that means you used a word whose reference was unclear. The common offenders are "it" and "this". These might refer to the previous noun, or a noun two or three nouns back, or the previous phrase, or the whole previous sentence. However, your readers shouldn't have to guess what you are referring to.

The official rule is that a referring word refers to the most recent noun or noun phrase it can. If that's not the noun you mean, you need to identify the referent by name. In general, "it" and "this" get used a lot as a crutch. If you find yourself using them, check to make sure you haven't abused them. If you find you have used "this" to refer to the whole previous sentence, use a more precise phrase instead. For instance, don't write

> Most people don't understand what mathematicians mean by induction. This is a difficulty.

Does "this" refer to "induction" or to the whole sentence? Instead write

> Most people don't understand what mathematicians mean by induction. This misunderstanding is a difficulty.

30. Revising Papers

This is tougher than you might think. If you have to add or move some definitions, or add or move a theorem, you will have to look over the whole paper to see what else is affected. Given the tight structure of good mathematics writing, a change in one place can and should require adjustments in many other places. Leave time for it!

31. Submitting your Finished Paper

You can always print out a copy and submit it by hand. This way you can look at it first and know exactly what I will see. However, if you want, you can submit your paper electronically. However, because different programs (the ones you use, the ones I use) handle math differently, there may be translation problems. So if you send me a paper electronically, please heed the following.

First, never send your paper to me by pasting it into an email message. Email sends only "ascii" (basically, roman letters); everything else can get translated or deleted. If you want to send a paper by email, use Eudora and make it an *attachment*. But it's much simpler to merely copy your paper into the drop box in the Math 9 folder on the classes server.

Second, I use MacWrite II, and this program cannot cope with the files of certain more recent word-processors. In particular, it cannot translate MacWritePro! (This is an intentional ploy by Apple to make MacWrite II obsolete.) Before you send me a document, use *your* word-processor to translate your document into MacWrite II or Word (4.0) form.

32. Questions? Comments?

If you have any questions or comments on the above, don't hesitate to check with me. If your question/comment is a good one (and most of them are), I will send it and my reply to all your classmates via computer (unless you ask me to keep your inquiry and my response private).

Finally, my writing isn't perfect either. If you find fault with the writing above, please point this out to me. We can all benefit from each other's criticism.

Reference

1. J. J. Price, Learning Mathematics Through Writing: Some Guidelines, *College Mathematics Journal* 20 (1989) 393–401.