Advice for Undergraduates on Special Aspects of Writing Mathematics

Abstract

There are several guides to good mathematical writing for professionals, but few for undergraduates. Yet undergraduates who write mathematics papers need special guidance. For instance, professionals may need help writing clear definitions, but at least they know why explicit definitions are needed and know the basic format. In general, mathematics has many special formats that are not mere technical conventions but instead serve important purposes. Often students are not conscious of these conventions, and they rarely know their purposes. This paper contains a guide for students written in light of these observations.

Interest in good mathematics writing — both by professors and by students — is on the increase. This interest has led to two recent publications on how to write mathematics, Gillman [3] and Knuth et al. [5], that nicely complement the classics in the field, such as the AMS Manual for Authors [1] and Steenrod et al. [9].

But all of these publications are aimed at professional mathematicians. Undergraduate students need other advice because they lack background knowledge presupposed in the guides for professionals. For instance, while some of these guides discuss how to state definitions effectively, none of them discuss why definitions are needed at all or what should be defined. Working mathematicians know answers to these questions, but undergraduate students often don't.

So students need mathematics writing guides, too. There are now many essays about student writing (see the compilations [2] and [10]), but these are discussions for teachers about different types of writing programs. I know of just one published how-to guide for students, the article [7] by Price. I highly recommend it.

However, [7] is not sufficient, because it is geared toward just one of the many different types of student writing assignments — the traditional assignment of writing up solutions to homework. Much more difficult is the sort of mathematics writing we professionals most often do, articles and reports. Until recently, few mathematics students were asked to write such papers, except perhaps senior majors, but more recently paper writing has become more common at all levels. Now that mathematics courses often have associated (computer) labs, one can also assign lab reports [4]. Finally, all of the types of writing described so far are "formal", in that (ideally) they are polished end-products giving the final state of a student's understanding. The current emphasis in writing instruction is to favor "informal", "expressive" writing — frequent, short, unpolished writing that accompanies and thus helps actively engage the student in the *process* of coming to understand the mathematics at hand. See the introductory article by Connolly in [2].

The situation at Swarthmore requires me to be particularly concerned about traditional papers. By faculty vote in 1985, all Swarthmore departments must give at least one freshman/sophomore Primary Distribution Course (PDC), and a student must take six PDCs (over two years) in different departments. Among other things, PDCs emphasize writing, and students are asked to write full-fledged papers in the style of the course's discipline. A full description and rationale are given in [6]; suffice it to say here that the writing is not only intended to help students comprehend the course material but also to give them a broad perspective on different writing styles.

Because I knew of no guides for undergraduates writing mathematics papers, I wrote my own, one for each PDC I gave, and I revise them each year as I see additional writing problems in the student papers I read. What follows is basically the latest version of my guide for a calculus course, with items specific to Swarthmore deleted.

I will be pleased if readers make use of this publication with their students, and PRIMUS has allowed me to state that faculty may reproduce this article for that purpose without requesting permission. If instead this article gives you ideas for producing your own handout — perhaps with some advice directly contrary to mine — that's fine too. In fact, I imagine many faculty have produced handouts for students on writing mathematics. I hope readers will share them with me; perhaps others will be published and a bibliography can be produced. The one other substantial essay of this sort I already know about is by Steen [8] and was written for mathematics majors.

Some readers of earlier versions of this paper have been critical. They felt it addressed the wrong issues for its student audience. Specifically, they felt: 1) My paper is about formal writing, but informal writing done in the process of learning is much more effective for students. 2) Even if one does assign formal end-product writing, my article says too little about big issues (e.g., how to plan out what to write) and too much about arbitrary technical conventions that only professionals need to learn.

I have thought about these criticisms, and they convinced me to delete some material. But mostly I decided that I disagree and thus need to say why, both in this preface and in the student guide itself. I have revised accordingly.

To respond to 1), informal process writing is surely very effective with many students and deserves wide use. If the only goal is to get students to learn mathematics (hence the phrase "writing to learn"), informal writing may be sufficient. However, if another goal is to help students communicate their newly gained knowledge to others, then writing with timetested conventions is called for. And if the goal is to help students develop writing skills that work in a variety of circumstances (learning to write as well as writing to learn), then hands-on experience with conventions of formal writing in several disciplines provides benefits that informal writing cannot.

As for 2), my essay does indeed devote much space to technicalities, but I contend that technical conventions are sometimes far from arbitrary and we (students and faculty) have a lot to learn by reflecting on them. The fact that some mathematics conventions have been universally adopted around the world suggests that they accomplish something important. I've thought about this (necessary because guides like [1,3,5,9] rarely give underlying reasons for writing conventions), and it seems to me that many mathematics conventions are crucial in making the always complicated progression of thought in mathematics papers much easier to follow.

In any event, those mathematics students who do have to write papers need certain advice which does not seem to be readily available. I am trying to fill this gap. So, in my handouts I concentrate on aspects of writing that are special to mathematics and for which I feel there are good reasons. I state the reasons I perceive. I'd be very interested in reader reaction. For instance, do you agree that the technicalities I talk about are important? What do you think of my discussion of the difference in "philosophy of references" between mathematics and other humanities?

Special Aspects of Writing Mathematics Papers

1. Introduction

Mathematics writing is different from ordinary writing and harder — in addition to all the requirement of ordinary good writing, there are additional constraints and conventions in mathematics. An additional constraint is that mathematics follows much more demanding rules of logic than ordinary discourse, and you must make your logic clear. Some of the additional conventions are those for defining new concepts and those for organizing the material through theorems and examples. This essay is about these constraints and conventions, especially the conventions.

Although you have seen these constraints and conventions in the mathematics texts you have read over the years, you have probably not realized when and why they should be followed. They may seem highly technical and arbitrary, and thus not worth learning if mathematics is a side show for you. Why, for instance, should calculations be displayed in just certain ways, or bibliographies be organized is just certain ways? Indeed, as far as I can tell some mathematics conventions *are* arbitrary, for instance, the ones for bibliographies; I won't discuss them further or hold you to them when you write papers. But many of the conventions, including those for displaying calculations, are not arbitrary. There are good reasons behind them, and once you understand these reasons, you understand the nature of mathematics a little better and you become more perceptive about how to explain things in ways that you can carry over to your other writing.

The organization of this essay is simple. Each section discusses a special aspect of mathematics writing that undergraduates have had trouble with. I try to tell you not only what you should do but also why.

It is important to understand what this essay is *not* about. It does not attempt to give you general advice about writing that will help in any field. It does not even advise you (except incidentally) about mathematics writing "in the large", i.e., how to come to grips with your topic and outline your approach. It is obvious that advice on those two topics is useful. It is probably not so obvious that advice on mathematics conventions is useful; that's why I've written this.

Mathematics conventions are not rigid rules, and some authors write very well although (perhaps because) they break them. But before you can decide intelligently to disregard a convention, you have to know what the convention is, know why it is, and have some practice following it.

2. What Kind of Mathematics Paper?

You will be writing *expository* mathematics papers. You will start with some issue (e.g., what is instantaneous change, how can derivatives be computed easily, how to optimize), show that there is some definition, theorem and/or computation method that precisely and correctly resolves the issue, and give various examples. Most papers written by working mathematicians are *research* papers; these present new discoveries and are usually written in a terse definition-theorem-proof style with limited intervening commentary. Expository papers are more informal, with much more discussion, but they still should be carefully and visibly organized — carefully because that's the way mathematics is; visibly because mathematics is subtle, so readers need many guideposts to follow it. Ideas are made very precise and expressed in symbols as well as words. Key definitions, theorems, examples and formulas are highlighted, numbered and referred to by number.

Your text is probably a pretty good example of expository style — ask your professor. For other examples, there should be some expository mathematics journals in the library. I recommend you browse at *The College Mathematics Journal, Mathematics Magazine* and *The American Mathematical Monthly.*

3. Know Your Reader

In *any* writing it is always important to ask yourself: For whom am I writing? In mathematics especially, the amount of explanation called for varies greatly with the reader's background. Assume you are writing for a student who has the same background as you, except that this student does not know the particular topic in your paper. Thus points that have been hard for you will be hard for your reader, and you should explain them carefully, using whatever approach finally worked for you. Just throwing down some cryptic calculations won't work!

Another important question is: How will my reader *use* my paper? If your reader really wants to learn the topic from you, he or she will frequently have to refer back to earlier definitions, theorems, explanations and examples — people have to go over mathematical items several times before they sink in. Therefore, key elements of your paper should be marked so that they can be found easily. This requires highlighting and numbering, which we will discuss later.

4. Titles

Every paper should have one. It should be informative without being too long. Choosing a good title in a mathematics paper is not so easy. Often a paper hinges on a concept that is defined only within the paper itself, so using use the name of that concept in the title will convey no meaning at all.

5. Introduction

Again, every paper should have one. In short papers it need only be a paragraph. Mathematics papers are hard to read, and for encouragement your reader deserves to be informed of what she is getting into and why she should care. Giving a good introduction is difficult in the same way giving a good title is: a satisfactory explanation may hinge on concepts you haven't introduced yet. But in a paragraph you can give a rough idea of the key concepts (admit that it's rough) and what you will do with them.

6. Division into Sections

Break papers of longer than 4 or 5 pages into sections. Your paper will have longer and fewer sections than this essay, because your paper will have a single topic. In a paper with only a few sections, the introduction should probably include a brief description of what each section will cover. At the start of each section remind the reader what the section is about. How does it fit into your scheme of things? Why is it there at all? Like the introductory section, these reminders will give the reader encouragement and guideposts. They will also help you. Writing them will force you to think hard about (and then maybe revise!) what you are trying to do and the order in which you are trying to do it.

7. Theorems

Any theorem in your paper should be highlighted — I suggest indenting it or putting extra vertical space around it. For instance,

> **Theorem.** For any constant p, the function $f(x) = x^p$ is differentiable, and
>
> $$\frac{d}{dx}\,x^p \;=\; px^{p-1}.$$

Other formats are allowed; browse and see. Why highlight theorems (and other key items such as examples)? Because mathematical arguments can be so complex. Formal statements of theorems (and other key items) serve as touchstones, like having an outline within the text body. They also makes it easy for the reader to refer back to key specifics. As further touchstones, if you have any proofs in your paper, you should indicate clearly where each proof begins and ends.

Levels of Confirmation (prove, verify, show, illustrate). In ordinary discourse, to prove something means to give any sort of fairly convincing evidence. In mathematics, "prove" has a much stricter meaning; you have proved something only if you have given a airtight argument — airtight because it harks back to definitions or to other theorems that hark back to definitions. Lesser levels of confirmation are often quite useful, but they are referred to with different words such as "show" and "illustrate".

For instance, consider the constant multiple rule in calculus: $(cf)' = cf'$. If you state this rule and then say "For instance, $(d/dx)4x^2 = 4(d/dx)x^2$, you have merely *illustrated* how to *use* the rule. If you actually prove from the definition that $(d/dx)4x^2 = 4(d/dx)x^2$, you have shown how the proof goes by proving a special case. Similarly, if you state the power rule $(d/dx)x^p = px^{p-1}$ and then prove $(d/dx)x^2 = 2x$, you have illustrated the rule by proving a special case (and this time the special case doesn't really suggest how the complete proof would go).

Mathematicians use "verify" somewhat like "prove". If you say you are going to verify a theorem, then you must prove it. But you can also verify a numerical claim; this is a much less demanding type of verification. Suppose I claim that, if my distance at time t is $s(t) = t^3$, then my speed at $t = 2$ is 12. I can verify this by the following simple computation: $s'(t) = 3t^2$, so $s'(2) = 3 \cdot 4 = 12$. Such a computation would not be called a proof. You could say "I have *shown* that the velocity is 12".

Suppose you give an example of a definition. For instance, suppose you have just given the definition of derivative and now use that definition to find that $(d/dx)x^2 = 2x$. You have not proved the definition, because definitions are conventions and cannot be proved (see next section). Rather, you have illustrated the definition by proving the formula for $(d/dx)x^2$.

In summary: "prove" has a very strict meaning; "show" is looser, and "illustrate" is looser still and refers to examples. Although illustration is the loosest, it is very important. Sometimes good examples will do more to help the reader understand and believe a result than a complete proof will.

8. Definitions

Definitions are a major way that mathematics writing differs from general writing. Most disciplines don't need to make definitions explicit nearly so often as mathematics does — they don't need to be so precise nor do they deal so regularly with situations outside common experience. Mathematical writing involves defining both words (e.g., derivative) and notation ($\frac{d}{dx}$). Notation is important because, if you use a new concept frequently, you need a shorthand way to refer to it or you will tie yourself in verbal knots.

When you give a definition, you can do it "in-line" (within a paragraph), but the word or phrase being defined should be highlighted, by <u>underlining</u>, by *italic print*, or by **boldface**. Boldface is now the most common format, since underlining is not common in typeset material and italic already has a use in both mathematical and ordinary writing to indicate *emphasis*. Here's an example of a definition: "A **prime number** is a positive integer with no positive integer divisors other than 1 and itself." Another format is to display definitions just like theorems are displayed. The display format should be reserved for the most important definitions. For instance, in calculus the definition of derivative might be displayed, but the definition of polynomial (if you need it at all) can be done in-line.

Once you define a word w, there are two things you shouldn't do. First, if another word w' is a synonym of w in ordinary English, don't use w' as if it has the same precise mathematical meaning as w. Second, don't use w itself in a loose ordinary English way that is in disagreement with its precise mathematical definition. As an example of the first rule, if you have defined "slope" at a point on the graph of f to mean the value of the derivative, don't afterwards sometimes say "steepness" when you mean derivative (unless you have explicitly said that "steepness" means the same as "slope"). As an example of the second rule, if you have defined "critical point" to be one at which $f'(x) = 0$, don't afterwards say "points where concavity changes are critical to graphing a curve". This would be perfectly good ordinary English, with "critical" having its general meaning of important, but it's not acceptable in your paper because "critical" has become a defined term with a more narrow meaning.

Why these two special restrictions? The whole point of mathematics definitions is to give ordinary words extra precision needed for a mathematical discussion. But there may be several different precise meanings that are mathematically useful and that one can attach to the same ordinary concept. An experienced mathematical reader, upon seeing "steepness" in your paper after you have defined "slope", will assume you are using both words because you need two different refinements of the ordinary slope concept. (Maybe you are using "steepness" to mean average slope.) Such a reader will then attempt to find where you defined steepness and will get frustrated. Similarly, once the reader has gotten straight the restricted meaning you have given words, he will get very confused if you use those words in ways that contradict your definitions.

Local Definitions. So far I have been talking about "global" definitions, those that apply throughout your paper. Most global definitions introduce words or notation. In contrast, local definitions apply only briefly, say, to the current example or current paragraph. Usually local definitions are definitions of symbols, for instance, "Let $f(x) = x^2$." A few paragraphs later it is acceptable to say "Now let $f(x) = e^x$." Local definitions don't require special highlighting, although they should be displayed if they involve complicated formulas (see Section 14, Displays). In any event, the item being defined should always come on the left-hand side of the equation; e.g., "Let $x^2 = f(x)$" is *wrong*. Why? Because when an equal sign is used with "let" to make a definition, the equal sign has a special meaning: the quantity on the right is being assigned to the quantity on the left. This is quite different from regular equality, where it makes no difference if you write $a = b$ or $b = a$.

9. Examples

Examples really help to make abstract concepts clear, so a good expository paper contains many – more examples than definitions and theorems. Examples are like definitions, in that they can appear in-line or be highlighted by indentation and extra spacing. For a very brief example, the in-line method is fine. However, a lengthier example should be displayed and numbered, especially if it is a key item of your work. Not only does this format draw attention to your example, but is also makes it easy to refer back to later (e.g., you can say see Example 3). If an example is a sample problem, make clear where the solution begins and ends.

10. Figures

Figures can be extremely helpful in an expository paper, just as they are in books. Each figure should be numbered (for easy reference later) and inserted shortly after the first reference to it. (Another convention is to put all figures at the end. If you use this alternative, please say so the first time you reference a figure.) Usually, each figure should have a caption as well; e.g., "The steeper the line, the greater the slope". If you know how to use computer software to produce a good figure and place it into your paper, great, but it's fine if you insert your figures by hand. You'll probably need more space than you first think.

11. Big Little Words (let, thus, so)

The three words above are common words in ordinary English, and many people use them casually. But they are big words in mathematics because they set forth the logic of your argument. They have precise mathematical meanings and should be used properly.

I have already discussed "let" under definitions. "Let" sets forth a convention, usually temporary, usually for a symbol. A related but not mathematically synonymous word is "suppose". You could say "Suppose $f(x) = x^2$", but you really shouldn't. "Suppose" is best used for temporary hypotheses, not temporary definitions. For example, imagine you want to show that, if $f'(x)$ changes sign at $x = a$, then f has a local extremum at a. You could argue as follows:

> Suppose $f'(x) > 0$ for $x < a$. Then the graph of f is increasing to the left of a. Since f' changes sign at a, we have $f'(x) < 0$ for $x > a$. So the graph of f is decreasing to the right of a. Since the graph of f first increases and then decreases, it has a local *maximum* at a. On the other hand, suppose $f'(x) < 0$ for $x < a$. Then ... f has a local *minimum* at a.

The point is, to say $f'(x)$ changes sign at a involves two cases: it changes from positive to negative, or vice versa. Each case is argued separately. So we suppose one case and then the other. It wouldn't really be correct to say instead "Let $f'(x) > 0$ for $x < a$" because we don't really have control over the sign of $f'(x)$, and the word "let" means that the matter is a convention that is up to us.

As for "thus" and "so", they mean that the next sentence or clause is a logical consequence of the previous sentence or clause. Therefore, if the next sentence would still make sense and be true even if you had not included the previous sentence, then the next sentence may *not* begin with "thus" or "so".

For example, consider the sentence

Let $x = 1$ and $y = 2$, so $x + y = 3$.

Here the use of "so" is correct. But now consider

Let $f(t)$ be the temperature at time t. Thus $f'(t) = \lim_{h \to 0}[f(t+h) - f(t)]/h$.

Here the use of "thus" is incorrect, because the fact that $f(t)$ represents temperature has nothing to do with why $f'(t)$ is defined the way it is.

12. When to Give Credit

The conventions for giving credit to others are somewhat different in mathematics than in general academic writing. Briefly put, direct quotes and almost direct quotes must be credited, but paraphrase generally need not be. Furthermore, mathematics papers rarely include direct quotes. Thus, while a humanities paper is full of footnotes giving credit, in a mathematics paper the credits are less frequent (and they typically appear in text). This is all elaborated below. Pay close attention, since failure to give proper credit is plagiarism, a major academic sin.

Rightly or wrongly, mathematics is regarded as having an existence independent of the words used to describe it. Thus your text may describe the Chain Rule theorem in slightly different words than anyone else, but that doesn't give its authors any special credit. If you use the Chain Rule, and you learned about it from your text, you should *not* reference your text at the point where you introduce the Chain Rule — the authors would never claim credit for the Chain Rule themselves. This same principle applies to definitions; don't reference your text for the definition of derivative.

Note. Theorems have two sorts of names: descriptive names, such as "Chain Rule"; and sequential names, such as "Theorem 6" and "Limit Rule II". A text gives a descriptive name only if it is widely used by others, so you can use the name "Chain Rule" too. But sequential names are specific to an individual text. Thus, if you must talk about Limit Rule II in your text, you have to reference your text and give a page number — otherwise readers won't have a clue what rule you are referring to. But why make your readers look this up when you could simply state the rule in your paper?

The "independent existence" perspective also explains why mathematicians rarely quote each other directly — your original words are generally not perceived to have an advantage over my paraphrase. This attitude is quite different from that in humanities, where the nuances in somebody's verbalization of an idea may make all the difference.

The original *discoverers* of mathematical results *are* given credit, and if a result is fairly recent, the paper in which it is first published must be referenced. But all the theorems in your calculus course are classical (some of them over 300 years old). They are so much considered a common heritage that people's names are only occasionally associated with individual results, and references to original publications are never made (except in a history of mathematics paper).

I've discussed giving credit for definitions and theorems. In your papers you are more likely to use examples (for instance, a max-min problem) taken from or based on your text. Should these be credited? Again, the answer is “usually no”. If you use the exact words or even just the exact numbers of an example from some book, then (and only then) give credit; but in general you shouldn't be using problems verbatim anyway. Why? Because in order to write about an example with understanding you should internalize it and make it your own before using it. And if you have internalized it, you can recreate it, no doubt with different numbers.

Finally, if your whole line of thinking comes from one source, it *is* appropriate to give *one* reference to it. Somewhere near the beginning of your paper, say something like “In writing this paper, I have drawn heavily on Goldstein et al. [1]” (where Goldstein is reference 1 in your bibliography) or “Much of my material is adapted from lectures by Smith [2]”.

13. Complicated Mathematical Expressions

Publishers call mathematics penalty copy because it is so hard to set on the page. Mathematics involves (among other things) subscripts and superscripts, numerous styles of type, special and/or large symbols (the integral sign), and expressions that must be stacked vertically (fractions). Traditional publishing involves setting type across simple lines, and many mathematical features interrupt that flow.

Typewriters and most computer word processing systems also are set up to go across lines, so you will have the same problem publishers do. Even if you have special mathematics publishing software (e.g., the TEX program I use and this journal uses), such software takes a long time to learn. And you're not trying to get a job as a mathematics typesetter; you're just trying to learn how to write. So what should you do about complicated math? Answer: Write in by hand any (parts of a) mathematical expression that would take too long to do well using the typewriter or computer programs you know.

On the other hand, some aspects of setting mathematics are easy to master, and others can be avoided. Almost every word processing system allows easy access to subscripts and superscripts, and you should use them. The next two subsections tell you how to deal with two major "problems" in setting mathematics: styles of type and fractions.

Fonts for Math. A **font** is a style of type, e.g., roman or italic. In books and journals, letters representing mathematical quantities, such as a, y and $f(x)$, are set in italic, as just done, to distinguish them from ordinary roman text. (Some mathematics is set in other fonts, e.g., Greek letters for angles in trigonometry and boldface for vectors in linear algebra.) The reason for special fonts is that, without them, sometimes it would not be clear what is intended, a mathematical expression or an ordinary word. For instance, consider the sentence "So the answer is $4pm$." Only the italics makes it certain that we mean 4 times p times m instead of 4 o'clock. Similarly, without italics "a" could be confused with the English word "a".

Unfortunately, math italic is different from ordinary italic, because only the letters are slanted, not the punctuation (parentheses, brackets, exclamation points). So, unless you have math italic fonts (most word processors don't), you have to go back and forth from roman to italic even to write simple expressions. So, the best thing to do is avoid italics entirely, and use an alternative convention usually reserved for typewritten documents: put extra blank space around mathematical letters and expressions. For instance,

Consider the variables x and y in the equation y = 2x+3 for a line.

Whether you use italic or not, you have to think about spacing *within* mathematics expressions too. "Tight spacing" like $\Delta y{=}f(x{+}h){-}f(x)$ makes already complicated expressions even harder to grasp. "Wide spacing" like $\Delta y \; = \; f(x \, + \, h) \, - \, f(x)$ is much preferred — see the difference? My personal preference is for tight spacing within inner groups (e.g., expressions within parentheses) and wide spacing otherwise: $\Delta y = f(x{+}h) - f(x)$. The point is, mathematical expressions often have several layers; you can always use parentheses and brackets alone to indicate the layers, but sometimes the use of spacing as well makes reader comprehension easier.

Whatever font you use for a mathematical symbol, *make sure you use that font all the time.* For instance, if you are talking about velocity, do not use v sometimes and v other times. Why? Mathematics is font-sensitive. Trained readers expect the same letter in different fonts (or with different accent symbols) to mean different things. Examples of the same letter appearing in different fonts are not common in calculus papers, but they occurs frequently in other fields of mathematics. For instance, a statistics paper might well use x for a data value, $\mathbf{x}$ for a sequence of data values, and $\bar{x}$ for the average value of the data.

Mathematics is case-sensitive too, that is, upper and lower case mean different things. For instance, that statistics paper probably uses X for the "random variable" of which x is a value. So, what do you do if you want to start a sentence with a variable you have named a?

Answer: Don't write A. Start with lower case, or (better) rewrite the sentence. For instance, instead of starting "$a > 0$ because", start "Quantity a is positive because".

Caution: Some computer software for mathematics is *not* case-sensitive. For instance, most versions of BASIC will treat the variables B and b as the same.

Fractions. Fractions in "built-up" form like

$$\frac{y_2 - y_1}{x_2 - x_1} \tag{1}$$

are especially hard to produce without special software. About the best you can do is use three separate lines, for numerator, fraction bar, and denominator. Even then it will be hard to center the parts properly. (You might think that underlining when you type the numerator will take care of the bar, but try it for display (1) and see what happens.)

So what should you do? Within a paragraph, for simple fractions you can avoid the problem by using "shilling" style, that is, a/b instead of $\frac{a}{b}$. But as soon as the numerator or denominator involves sums and differences, the shilling approach requires parentheses. For instance, in shilling form display (1) must be written as $(y_2-y_1)\,/\,(x_2-x_1)$; why? If a fraction already involves parentheses in built-up form, it helps the reader if you use brackets on the outside in shilling form: $[f(x+h) - f(x)]/h$. Alas, if a fraction is sufficiently complicated, people find it almost impossible to make sense of shilling form, no matter what; consider

$$[f(x+h)(g(x+h) - g(x)) + g(x)(f(x+h) - f(x))]/[(x+h) - h].$$

Don't write such an expression in a paper, even though it is correct. In such cases you must use built-up form. Suggestion: draw in at least the fraction bar by hand.

Handwriting Mathematics. If an expression is going to take a lot of time to set (even with the advice above), then leave a blank space in your paper and write the whole thing in by hand. At first, you will probably grossly underestimate the amount of space you need for hand insertion. But if the rest of your paper is written on a computer, it will be easy to output another copy with more space.

14. Displays

Any long expression with mathematical symbols is **displayed** — centered on a line by itself, with extra vertical space around it. There are three reasons for displays. First, if an expression is particularly important, you draw attention to it by displaying it. Second, longer mathematical expressions tend to be tall. To fit something like $\displaystyle\int_2^5 \frac{dx}{x^2+1}$ in-line requires adding disconcerting interline space (see it?), or else making the symbols disconcertingly small, like this: $\int_2^5 \frac{dx}{x^2+1}$. Third, if you put the expression inside a paragraph, it might come at the end of a line and have to be broken across two lines. You can hyphenate words, but mathematical expressions are usually longer than words and don't bear line breaks well. So if an expression would have to be broken if it appeared in the middle of a paragraph, it should instead be put in a display. For instance, if you wish to say

$$y' = \frac{d}{dx}\,(x^2+3x)$$

within a line, it would be bad form to break this after the + and *terrible* form to break it after $\frac{d}{dx}$. According to some writers, it is even bad form to break it after "=".

A display should be numbered if you find that you refer to it anywhere other than within a few lines of it. The numbering can appear either on the left, as in

$$\text{(2)} \qquad y' = \frac{d}{dx}\,(x^2+3x),$$

or on the right, as in

$$y' = \frac{d}{dx}\,(x^2+3x), \tag{3}$$

but be consistent. Just below a display you can refer to it as the "previous display", but two pages later it's too long-winded to refer to "the third display on page 7". Besides, if you modify your paper, changing the page breaks, you will have to rewrite such a reference completely. That's why we use numbering.

Multiline Displays. In most cases, these should be lined up so that the main connectives (usually an =, but maybe $\leq$ or $\Longrightarrow$) line up vertically:

$$\begin{aligned} 2x+1 &= 2+3, \\ 2x &= 4, \\ x &= 2. \end{aligned} \tag{4}$$

Here, lining up the equal signs emphasizes that the same thing has been done to both sides in getting from one line to the next.

Similarly, one writes

$$\begin{aligned} x^2+2x &< x^2+2x+1 \\ &= (x{+}1)^2. \end{aligned} \tag{5}$$

Notice that display (4) has expressions on both sides of the equal sign on each line, but display (5) has only a right-hand side after the first line. When a display could have been written as one long line, e.g., $x^2+2x < x^2+2x+1 = (x+1)^2$, but is written in several lines for emphasis or because it won't all fit on one line, then this "right side only" format is appropriate. This format emphasizes that each expression (not each line) is obtained by doing something to the previous expression.

Notice the commas and the period in display (4). Mathematics is written in sentences, and this display is a sentence consisting of a sequence of clauses, the equations. Thus the clauses are separated by commas (some writers would use semicolons) until the sentence ends with a period. However, there is an alternative convention (followed by about half the publishers in the US) which leaves off punctuation at the end of displays. Personally, I'm not so concerned about the commas (the separation into different lines tells me to pause) *but I do feel strongly about the period*. Sometimes a sentence does *not* end at the end of the display; that is, I am supposed to keep reading in order to understand why the display makes sense. On the other hand, if the display ends in a period, that is a signal that I am supposed to figure out for myself why the display is legitimate. In other words, the presence or lack of a period at the end of a display tells me whether this is a point at which I need to stop and digest what has just been said. Since it is very important for the author to give the reader cues like that, the punctuation at the end of a display is crucial.

In display (5) there is no comma after the first line and there never should be. Why?

Explaining your displays. The fact that calculations *can* be written as one long display without words doesn't mean that they *should* be written this way. Mathematics does not consist of calculations alone! Very elementary algebra, as in (4), need not be explained, but more complicated calculations should be, especially if the calculations are justified by mathematics that the reader is just learning. There are two ways to provide the explanation. One way is to put it before, between and after the lines of calculation. For instance, suppose you have defined $f(x) = x^2$, and want to show that $[f(x+h)-f(x)]/h = 2x+h$. You could write:

> Substituting the definition $f(x) = x^2$,
>
> $$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h}.$$
>
> Then expanding the first square, combining like terms, and finally canceling the common factor h, we obtain
>
> $$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x^2 + 2xh + h^2) - x^2}{h} \\ &= \frac{2xh + h^2}{h} \\ &= 2x + h. \end{aligned}$$

An alternative approach is to put all the reasons in comments on the right:

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2-x^2}{h} && \text{[def. of } f(x)\text{]}\\
&= \frac{(x^2+2xh+h^2)-x^2}{h} && \text{[expand]}\\
&= \frac{2xh+h^2}{h} && \text{[combine terms]}\\
&= 2x+h. && \text{[divide]}
\end{aligned}$$

Actually, most of these steps are simple enough that you could skip the last two side comments. But to display the four lines of algebra just above without any explanation is unacceptable to me.

If a line in a commented display it quite long, put the comment one line below it:

$$\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right] = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]+\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$
$$\text{[Theorem on limit of a sum]}$$

It is best if your displays themselves have verbs, like "=". Compare the following.

Version 1:

And so we conclude that the derivative $f'(x)$ should be defined as

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\,. \tag{6}$$

Version 2:

And so we conclude that the derivative should be defined by

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\,. \tag{7}$$

Version 3:

And so we arrive at

Definition 1. The derivative of $f(x)$, denoted $f'(x)$, is

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\,. \tag{8}$$

Version 1 is poor, 2 is pretty good and 3 is best. Why? Because at a later point you may well want to remind readers of the definition. If you refer them back to display (6), they have got to go back into the text above the display to find out what the display is defining. If you refer them back to Eq. (7), the display itself is enough, unless they have forgotten what the symbolism $f'(x)$ means. If you refer them back to Eq. (8), they can't help but catch the boldface word "Definition" just above, and so they can't help but see that Eq. (8) is not just an equation but actually the definition.

15. Two Common Mistakes

The first mistake is not to rely on mathematical symbols enough. If, for instance, you refer to the same function more than once, give it a name (let $f(x) = x^2$). Even if you are just discussing functions in general, you need to give the generic function a name if you are going to say anything about it. (e.g., for any function $f(t)$, $f'(t)$ measures the rate at which…). More generally, if you are working towards the precise formula in some definition or theorem, it may be wise to state the formula early on, before you have even explained all the symbols that occur in it. This way at least you have something concrete to refer to as you work through your explanation. Otherwise you bog down in vague verbiage.

The second mistake is to rely on mathematical symbols too much. This happens when you present several lines of computation without any commentary.

16. Miscellaneous

Please number all your pages. This is especially helpful if you should staple them out of order (which happens) or if I get them out of order while reading (this also happens).

If you have any questions or comments on the above, don't hesitate to check with me. If your question/comment is a good one (and most of them are), I will send it and my reply to all your classmates via computer (unless you ask me to keep your inquiry and my response private).

Finally, my writing isn't perfect either. If you find fault with the writing above, please point this out to me. We can all benefit from each other's criticism.

References

1. American Mathematical Society, *A Manual for Authors of Mathematical Papers*, 8th ed., pamphlet, Providence, R.I., 1984.

2. P. Connolly and T. Vilardi, eds., *Writing to Learn Mathematics & Science* Teachers College Press, New York, 1989.

3. L. Gillman, *Writing Mathematics Well*, Mathematical Association of America, Washington, D.C., 1988.

4. G. D. Gopen and D. A. Smith, What's an Assignment Like You Doing in a Course Like This Writing to Learn Mathematics, *College Mathematics Journal* 21 (1990) 2–19, reprinted from [2].

5. D. E. Knuth, T Larrabee and P. M. Roberts, *Mathematical Writing*, MAA Notes #14, Mathematical Association of America, Washington, D.C., 1989.

6. S. B. Maurer, Writing in Mathematics at Swarthmore: PDCs, in [10].

7. J. J. Price, Learning Mathematics Through Writing: Some Guidelines, *College Mathematics Journal* 20 (1989) 393–401.

8. L. A. Steen, *Some Elementary Principles of Mathematical Exposition*, unpublished notes (revised version), St. Olaf College, May, 1973.

9. N. E. Steenrod, P. R. Halmos, M. M. Schiffer and J. E. Dieudonne, *How to Write Mathematics*, pamphlet, 3rd printing, American Mathematical Society, Providence, R.I., 1983.

10. A. Sterrett, ed., *Using Writing to Teach Mathematics*, Mathematical Association of America, Washington, D.C., 1990.

Biographical Sketch

Stephen Maurer (B.A. Swarthmore 1967, Ph.D. Princeton 1972) is a Professor of Mathematics at Swarthmore College. Previously he taught at the Phillips Exeter Academy (1969-73), the University of Waterloo in Ontario (73-74), and Princeton (74-79). His research has been in combinatorics, with forays (sometimes continuous) into mathematical biology, economics and anthropology. As for curricular activities, he has written and spoken widely on discrete mathematics. During 1982-84, he was a Program Officer at the Sloan Foundation, working primarily on quantitative education. From 1981 to 1987, he chaired the MAA committee on high school contests (AHSME, AIME, USAMO). His freshman-sophomore text, "Discrete Algorithmic Mathematics" (with Anthony Ralston) appeared in Fall 1990.